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Revisiting Sylvan's Semantics for S0.6° and related systems

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Abstract

In the present paper, we first investigate the logics obtained by considering frames with one or more kinds of non-normal worlds: *Sylvan frames* and *Sylvan logics*, as we call them. Determination theorems for all the logics obtained are proven, and their relation to classical systems of modal logic are discussed. In a second part, we consider the strict logics obtained by taking, for instance, the set of normal worlds as distinguished. We show, among other things, that Sylvan's claims about the semantics for **S0.6°** and **S0.7°**, besides those for **S0.9°** and **S1°**, are also inadequate, determining, in fact, stronger logics.

Keywords: Modal logics, strict classical logics.

1 Introduction

In his very interesting paper [5], Richard Sylvan intended to present relational semantics for all well-known systems of modal logic of Lewis, Lemmon and Feys'. The gist of Sylvan's semantics was to consider relational frames having, in addition to the normal worlds, one or more sets of non-normal ones, where modalized formulas are evaluated according to several different conditions. One well-known such condition is that all necessities are false and all possibilities true: here we have the "queer" worlds of Kripke's [3]. Other kinds of non-normal worlds Sylvan considered were the following:

* opposite: $\Box\alpha$ is true iff α is false in every accessible world;
* contrary: $\Box\alpha$ is true iff α is false in some accessible world;
* perverse: $\Box\alpha$ is true iff α is true in some accessible world;
* rafferty: $\Box\alpha$ is arbitrarily true or false.

By combining different kinds of non-normal worlds we obtain semantics for several non-normal modal logics. In particular, Sylvan contended that frames consisting of normal and opposite worlds would yield determination theorems for the logics **S1°** and **S1**, if we take the set of normal worlds to be distinguished. This, however, was not so. In his review of Sylvan's paper, M. J. Cresswell guessed (what he later proved in [2]) that Sylvan's semantics did not characterize **S1**, but a stronger system which Cresswell called $\mathbf{S1}^{+}$. In short, Cresswell concluded, Sylvan's proposed semantics for **S1°** and **S1** were inadequate ([2], p.39).

Nevertheless, the logics characterized by frames with non-normal worlds remain interesting in their own right. Our motivation to investigate them in this paper is mainly technical, with a dash of historical curiosity. However, being non-normal, these logics can find applications in the areas of epistemic and deontic logic (where usual rules such as necessitation are deemed too strong). The fact that truth-conditions for modalized formulas vary depending on the kind of world where the formulas are evaluated also suggests applications such as modelling contextual modals in linguistics. (But we won't pursue these matters here.)

Both Sylvan and Cresswell considered just a few such systems, and only frames where the set of normal worlds is distinguished. (Cresswell only discusses **S1°**, **S1** and $\mathbf{S1}^{+}$. Sylvan's original formulation, by the way, selects an actual world among the normal worlds of the frame.)

In the present paper, we first investigate the logics obtained by considering frames with one or more kinds of non-normal worlds: *Sylvan frames* and *Sylvan logics*, as we call them. Determination theorems for all the logics obtained are proven, and their relation to classical systems of modal logic are discussed. In a second part, we consider the strict logics obtained by taking, for instance, the set of normal worlds, or normal and perverse worlds, as distinguished. We show, among other things, that Sylvan's claims about the semantics for **S0.6°** and **S0.7°**, besides those for **S0.9°** and **S1°**, are also inadequate, determining, in fact, stronger logics.

2 Logics and frames

We will work in a basic modal language consisting of a countable set Φ of propositional variables, the propositional constant $\bot$, and the primitive operators $\to$ and $\Box$. The remaining standard operators $\neg$, $\Diamond$, $\wedge$, $\vee$, $\leftrightarrow$ and the constant $\top$ are defined in the usual way.

All logics considered in this paper will be extensions of classical propositional logic, so they include the set PL of all tautologies, as well as being closed under modus ponens (MP) and uniform substitution (US). Thus, for the purposes of this paper, a *logic* is a set of formulas that includes all tautologies and is closed under MP and US. Following [1], we say that a formula belonging to a logic is a *thesis* of that logic. If all instances of some schema belong to a logic, or if the logic is closed under some rule of inference, we say that it *has* or *provides* that schema or rule.

We start by characterizing our logics semantically, defining frames and models with several kinds of non-normal points (or worlds, states, indices), and checking which formulas turn out to be valid and which rules of inference hold.

Depending on the kinds of non-normal points we admit, we can have different frames: we may have only normal points, perhaps opposite and perverse ones, and so on. In the most general case, a *frame* $\mathfrak{F}$ would be a structure $\langle U, U_1, \ldots, U_n, \ldots, R\rangle$, where U is a nonempty set, the universe of of the frame, and the U_is are pairwise disjoint subsets of U such that $N = U_1$ is the set of normal points of the frame and all the others are non-normal. We also require that $\bigcup U_i = U$ and, of course, $R \subseteq U \times U$ is an accessibility relation.

Allowing different subsets of U to be empty will give us different logics. For instance, if all but N are empty, we have normal modal logics, and if only N and a set Q of "queer worlds" are nonempty, we have logics like **E2**° and its extensions.

In this paper, however, we will consider only frames with a nonempty set of normal points and one or more sets of non-normal ones which will include only opposite, contrary and perverse points (in short, *Sylvan frames*), these being the main kinds of worlds considered by Sylvan in his paper.[1] Thus:

Definition 2.1. *A* frame $\mathfrak{F}$ *is a structure* $\langle U, N, O, C, P, R\rangle$*, where* U *is a nonempty set, the universe of the frame;* N*,* C*,* O *and* P *are pairwise disjoint subsets of* U*, respectively, the set of normal, opposite, contrary and perverse points of the frame. We also require that* $U = N \cup O \cup C \cup P$*, and that* $N \neq \emptyset$*. Finally,* $R \subseteq U \times U$ *is an accessibility relation.*

Our definition of frame is also more general than Sylvan's original one, since we do not require his *critical condition*: that every point should be accessible to some normal one. (But we discuss frames with this condition later on.)

We can also call the frames above defined *nocp-frames*, since they have all kinds of non-normal points. But of course we can remove one or more kinds of such points, thus obtaining *noc-frames* (that is, we remove the perverse points or, what amounts to the same, require that the set P be empty), *ncp-frames* (without opposite points), *np-frames* (only normal and perverse) and so on, until we reach *n-frames*, which are just the ordinary relational frames in which all points are normal.[2]

We will use the names **nocp**, **noc**, **ncp** and so on for the logic of the corresponding class of frames; they are related as depicted in Figure 1. We have the weakest logic,

[1] Having rafferty points would of course result in having only classical tautologies as valid — unless such points were not distinguished.

[2] Notice that we are not imposing any restrictions on the accessibility relation R. We can do that, requiring that R is reflexive, or transitive, and so on, but this would be the subject of another paper.

nocp, on the left of the image, and **n** (which is just the smallest normal logic **K**) on the right.

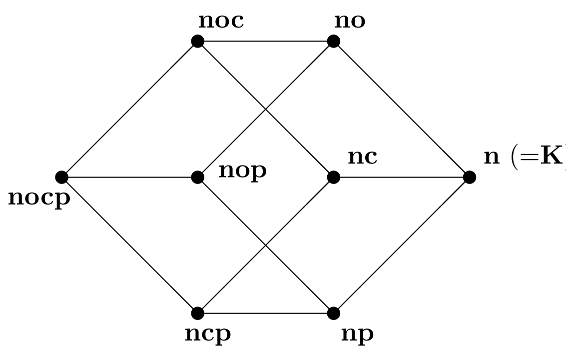

Figure 1: Sylvan logics

We will show later on that all these logics are indeed distinct. Before going into that, however, let us define models and truth — a general definition which will work for all logics investigated in this paper.

Definition 2.2. *Let $\mathfrak{F}$ be a frame, as above defined. Then $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ is a* model*, where V is a valuation, that is, a function from the set Φ of variables to $\mathcal{P}(U)$. We also say that $\mathfrak{M}$ is* based on $\mathfrak{F}$.

We have the usual definition of truth at a point in a model, except for modal clauses, where the truth condition will depend on whether a point is normal or non-normal.

Definition 2.3. *Let $\mathfrak{M} = \langle U, N, O, C, P, R, V \rangle$ be a model, and $x \in U$. Then:*

(a) $\mathfrak{M}, x \Vdash \mathbf{p}$ *iff* $x \in V(\mathbf{p})$, *for* $\mathbf{p} \in \Phi$;

(b) $\mathfrak{M}, x \Vdash \bot$ *never;*

(c) $\mathfrak{M}, x \Vdash \alpha \to \beta$ *iff* $\mathfrak{M}, x \nVdash \alpha$ *or* $\mathfrak{M}, x \Vdash \beta$;

(d) $\mathfrak{M}, x \Vdash \Box\alpha$ *iff*

1. $x \in N$ *and, for every* y *such that* Rxy, $\mathfrak{M}, y \Vdash \alpha$, *or*
2. $x \in O$ *and, for every* y *such that* Rxy, $\mathfrak{M}, y \nVdash \alpha$, *or*
3. $x \in C$ *and there is some* y *such that* Rxy *and* $\mathfrak{M}, y \nVdash \alpha$, *or*
4. $x \in P$ *and there is some* y *such that* Rxy *and* $\mathfrak{M}, y \Vdash \alpha$.

Obviously, the truth conditions for formulas with the possibility operator are the following:

(e) $\mathfrak{M}, x \Vdash \Diamond\alpha$ iff

1. $x \in N$ and there is some y such that Rxy and $\mathfrak{M}, y \Vdash \alpha$, or
2. $x \in O$ and there is some y such that Rxy and $\mathfrak{M}, y \nVdash \alpha$, or
3. $x \in C$ and, for every y such that Rxy, $\mathfrak{M}, y \nVdash \alpha$, or
4. $x \in P$ and, for every y such that Rxy, $\mathfrak{M}, y \Vdash \alpha$.

Definition 2.4. *A formula α is* true *in a model $\mathfrak{M}$ ($\mathfrak{M} \Vdash \alpha$) if it is true in every state of the model, that is, if $\mathfrak{M}, x \Vdash \alpha$ for every x in $\mathfrak{M}$. We say that α is* satisfiable *in a model if true at some state of the model;* falsifiable *if its negation is satisfiable.*

Definition 2.5. *A formula α is* valid at a state x *in a frame $\mathfrak{F}$ ($\mathfrak{F}, x \Vdash \alpha$) if α is true at x in every model based on $\mathfrak{F}$. We say that α is* valid in a frame $\mathfrak{F}$ *($\mathfrak{F} \Vdash \alpha$) if it is valid at every state in $\mathfrak{F}$, and* valid in a class C of frames *($\Vdash_{\mathsf{C}} \alpha$) if it is valid in every frame in C.*

Given these definitions, the following proposition is easily proved.

Proposition 2.6. *Let C be any class of frames. Then:*

(i) *if α is a tautological consequence of $\alpha_1, \dots, \alpha_m$ $(m \geq 0)$, and $\Vdash_{\mathsf{C}} \alpha_1$, ..., $\Vdash_{\mathsf{C}} \alpha_m$, then $\Vdash_{\mathsf{C}} \alpha$;*

(ii) *if $\Vdash_{\mathsf{C}} \alpha \leftrightarrow \beta$, then $\Vdash_{\mathsf{C}} \Box\alpha \leftrightarrow \Box\beta$.*

Proof. The proof of (i) is straightforward. As for (ii), suppose that $\alpha \leftrightarrow \beta$ is valid in a class C of frames, but $\Box\alpha \leftrightarrow \Box\beta$ is not. Thus, it fails at some state x of some model based on some frame in C; say, at $x \Vdash \Box\alpha$ but $x \nVdash \Box\beta$. We have four cases. (a) If x is normal and $x \nVdash \Box\beta$, there is some y such that Rxy and $y \nVdash \beta$. But since we also have $y \Vdash \alpha$ (because $x \Vdash \Box\alpha$), it follows that $y \nVdash \alpha \leftrightarrow \beta$, against the hypothesis that $\alpha \leftrightarrow \beta$ is valid in C. (b) If x is opposite, we have, since $x \nVdash \Box\beta$, that there is some y such that Rxy and $y \Vdash \beta$. But since we also have $y \nVdash \alpha$ (because $x \Vdash \Box\alpha$), it follows that $y \nVdash \alpha \leftrightarrow \beta$, against the hypothesis. (c) and (d) are analogous. ■

Recall that a modal logic is classical if it provides the following rule:

(RE) $\alpha \leftrightarrow \beta \,/\, \Box\alpha \leftrightarrow \Box\beta$.

Thus, from the preceding proposition we gather that our logics are classical modal logics.

Let us now consider Sylvan's critical condition, that is, every point in a frame is accessible to some normal one. Let us call a formula *cc-valid* if it is valid in the class of all frames satisfying the critical condition. We can show the following result.

Proposition 2.7. *α is valid iff α is cc-valid.*

Proof. If α is valid then it is obviously cc-valid, so suppose that α is not valid. Thus, there is some frame $\mathfrak{F} = \langle U, N, O, C, P, R \rangle$, some model $\mathfrak{M}$ based on this frame, and some $x \in U$ such that $\mathfrak{M}, x \nVdash \alpha$. Suppose $\mathfrak{F}$ does not satisfy the critical condition. We define another frame $\mathfrak{F}' = \langle U', N', O, C, P, R' \rangle$, where, for some $z \notin U$, $U' = U \cup \{z\}$ and $N' = N \cup \{z\}$ (z is thus a new normal point), and such that $R' = R \cup \{\langle z, y \rangle \mid \text{ for all } y \in U'\}$. This new frame obviously satisfies the critical condition, because every point is accessible to z. We now define another model $\mathfrak{M}'$, based of $\mathfrak{F}'$, such that $V'(\mathbf{p}) = V(\mathbf{p})$, for every $\mathbf{p} \in \Phi$. $\mathfrak{M}'$ is what we call a safe extension of $\mathfrak{M}$, and it is straightforward to show that, for every $y \in U$ and every formula β, $\mathfrak{M}, y \Vdash \beta$ if and only if $\mathfrak{M}', y \Vdash \beta$. It follows that $\mathfrak{M}', x \nVdash \alpha$, so α is not cc-valid. ■

The argument above will hold for ncp-frames and so on. In particular, it holds because we are not putting any restrictions (like transitivity, symmetry and so on) on the accessibility relation. We can show this will hold for reflexive, serial and transitive frames also, but trouble arises with conditions such as symmetry and euclideanity.

Sylvan's motivation to require the critical condition was to ensure that the logics provide the rules $\Box\alpha \,/\, \alpha$ and $\Box\top \to \Box\alpha \,/\, \alpha$, but more on this further on.

3 The basic logic nocp

In this section we will consider the smallest of our Sylvan logics, namely, **nocp**, the logic of the class of frames having all kinds of points: nocp-frames, in short. We will present it in a little more detail, since the other logics will be extensions of it, and small changes in the proofs will easily yield determination theorems for the other systems.

As we saw before, **nocp** is a *classical* modal logic (because it has RE); in fact we will show that it properly extends the smallest classical modal system, **E**. A bit of calculation from the semantics shows us that **nocp** provides the axiom schemes and rules of inference listed in Table 1 and Lemma 3.1 below. Names of formulas are mostly standard. We also follow Chellas and Segerberg's naming procedure (in [1]) for schemata: where N names the formula $\Box\top$, Q names $\Diamond\bot$, and S is some schema, nS, qS and $\Box$S denote, respectively, $\Box\top \to$ (S), $\Diamond\bot \to$ (S), and $\Box$(S).

Well-known axioms are M, C, F, K, and X. Table 1 below shows which of them hold in which kinds of points, no matter what properties the accessibility relation may have. We also added some new schemas (W, V, and so on) which will be important with regard to some non-normal points.

As we can see, none of the above schemes are valid if all kinds of points are considered. What about their n- and q-versions? As we gather from Table 2, nM, nV, nY, qW, qC, qK and qA are valid in all points — and are thus provided by **nocp** and all other Sylvan logics. Notice that axiom N, that is $\Box\top$, is true at any normal point of any model. However, if the accessibility relation R is serial (there always is an accessible

	Axioms	N	O	C	P
M	$\Box(\alpha \wedge \beta) \to (\Box\alpha \wedge \Box\beta)$	✓	-	-	✓
C	$(\Box\alpha \wedge \Box\beta) \to \Box(\alpha \wedge \beta)$	✓	✓	✓	-
F	$\Box(\alpha \wedge \beta) \leftrightarrow (\Box\alpha \wedge \Box\beta)$	✓	-	-	-
K	$\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$	✓	✓	✓	-
X	$(\Box(\alpha \to \beta) \wedge \Box(\beta \to \gamma)) \to \Box(\alpha \to \gamma)$	✓	✓	-	-
W	$\Box(\alpha \vee \beta) \to (\Box\alpha \wedge \Box\beta)$	-	✓	✓	-
V	$(\Box\alpha \wedge \Box\beta) \to \Box(\alpha \vee \beta)$	✓	✓	-	✓
Z	$\Box(\alpha \vee \beta) \leftrightarrow (\Box\alpha \wedge \Box\beta)$	-	✓	-	-
Y	$\Box(\alpha \wedge \beta) \to (\Box\alpha \vee \Box\beta)$	✓	-	✓	✓
A	$(\Box\alpha \vee \Box\beta) \to \Box(\alpha \wedge \beta)$	-	✓	✓	-

Table 1: Some axioms and points in which they hold

point y to any point x), then N holds also at perverse points. Analogously, axiom Q, that is, $\Diamond\bot$, is true at any opposite point, and, if R is serial, it will also hold at contrary ones.

	N	O	C	P
nM	✓	✓	✓	✓
nC	✓	✓	✓	-
nF	✓	✓	✓	-
nK	✓	✓	✓	-
nX	✓	✓	✓	-
nW	-	✓	✓	-
nV	✓	✓	✓	✓
nZ	-	✓	✓	-
nY	✓	✓	✓	✓
nA	-	✓	✓	-

	N	O	C	P
qM	✓	-	-	✓
qC	✓	✓	✓	✓
qF	✓	-	-	✓
qK	✓	✓	✓	✓
qX	✓	✓	-	✓
qW	✓	✓	✓	✓
qV	✓	✓	-	✓
qZ	✓	✓	-	✓
qY	✓	-	✓	✓
qA	✓	✓	✓	✓

Table 2: n- and q-versions

Lemma 3.1. *The following rules of inference are provided by* **nocp** *and its extensions:*

(RnM) $\alpha \to \beta \; / \; \Box\top \to (\Box\alpha \to \Box\beta)$,

(RqW) $\alpha \to \beta \; / \; \Diamond\bot \to (\Box\beta \to \Box\alpha)$,

(RRE) $\alpha \leftrightarrow \beta \; / \; \gamma \leftrightarrow \gamma[\alpha/\beta]$,

(RnN*) $\alpha \; / \; \Box\top \to \Box\alpha$, (RnN$_*$) $\Box\top \to \Box\alpha \;/\; \alpha$,

(RN$_*$) $\Box\alpha \;/\; \alpha$.

Proof. The proof involves a bit of calculation, but is actually quite straightforward. We will show RN$_*$ to illustrate the use of frames satisfying the critical condition. So suppose α is not valid. Then it is not cc-valid and, for some model based on a frame satisfying the critical condition, and some point x in it, $x \not\Vdash \alpha$. But then there is some normal point z such that Rzx, from what it follows that $z \not\Vdash \Box\alpha$, so $\Box\alpha$ is not valid. ■

We axiomatize **nocp** by adding the following axioms and the rule RE to classical propositional logic:

(nM) $\Box\top \to (\Box(\alpha \wedge \beta) \to (\Box\alpha \wedge \Box\beta))$,

(qW) $\Diamond\bot \to (\Box(\alpha \vee \beta) \to (\Box\alpha \wedge \Box\beta))$,

(nNP) $\Box\top \to [((\Box\alpha \wedge \Box\beta) \to \Box(\alpha \wedge \beta)) \vee (\Box(\gamma \vee \delta) \to (\Box\gamma \vee \Box\delta))]$,

(qOC) $\Diamond\bot \to [((\Box\alpha \wedge \Box\beta) \to \Box(\alpha \vee \beta)) \vee (\Box(\gamma \wedge \delta) \to (\Box\gamma \vee \Box\delta))]$,

(RE) $\alpha \leftrightarrow \beta \;/\; \Box\alpha \leftrightarrow \Box\beta$.

As we see, **nocp** is a *classical* modal logic, because it provides RE. However, since neither nM nor qW are valid in **E** (as a simple semantic argument will show), **nocp** is a proper extension of **E**.

Instead of nM, qW and RE we could have used RnM and RqW. This is established via the following proposition, whose proof is straightforward:

Lemma 3.2. *(i) A modal logic provides nM and RE iff it provides RnM. (ii) A modal logic provides qW and RE iff it provides RqW.*

We actually have two classes of points in a nocp-frame: normal and perverse, on the one side, where $\Box\top \vee \Diamond\top$ holds, and opposite and contrary, on the other side, where $\Box\bot \vee \Diamond\bot$ holds. Of course neither of these formulas is a thesis of **nocp**, whose frames have all kinds of points. Axioms nNP and qOC are needed to distinguish, on the one hand, normal and perverse point; on the other, opposite and contrary ones. They are used to show that **nocp** has the two inference rules below, RvC$_n$ and RcP$_n$, which will be needed in proving completeness.

Lemma 3.3. nocp *provides the following two rules:*

(RvC$_n$) $$\frac{(\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta}{(\Diamond\bot \wedge \Box\gamma \wedge \Box\delta \wedge \neg\Box(\gamma \vee \delta) \wedge \Box\beta) \to (\Box\alpha_1 \vee \ldots \vee \Box\alpha_n)}, \text{ for } n \geq 0,$$

(RcP$_n$) $$\frac{\alpha \to (\beta_1 \vee \ldots \vee \beta_n)}{(\Box\top \wedge \Box\gamma \wedge \Box\delta \wedge \neg\Box(\gamma \wedge \delta) \wedge \Box\alpha) \to (\Box\beta_1 \vee \ldots \vee \Box\beta_n)}, \text{ for } n \geq 0.$$

Proof. (i) For RvC_n, suppose $n = 1$, and that $\vdash \alpha \to \beta$. By RqW we have $\vdash \Diamond\bot \to \Box\beta \to \Box\alpha$, which is equivalent to $\vdash (\Diamond\bot \wedge \Box\beta) \to \Box\alpha$, from which we get, by propositional logic,

$$\vdash (\Diamond\bot \wedge \Box\gamma \wedge \Box\delta \wedge \neg\Box(\gamma \vee \delta) \wedge \Box\beta) \to \Box\alpha.$$

Let now $n > 1$, and suppose we have the rule for $n-1$. Suppose now that $\vdash (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta$, that is, $\vdash (\alpha_1 \wedge \ldots \wedge (\alpha_{n-1} \wedge \alpha_n)) \to \beta$. Then, by RvC_{n-1}, we get

$$\vdash (\Diamond\bot \wedge \Box\gamma \wedge \Box\delta \wedge \neg\Box(\gamma \vee \delta) \wedge \Box\beta) \to (\Box\alpha_1 \vee \ldots \vee \Box(\alpha_{n-1} \wedge \alpha_n)).$$

Consider now the following instance of qOC:

$$\Diamond\bot \to [((\Box\gamma \wedge \Box\delta) \to \Box(\gamma \vee \delta)) \vee (\Box(\alpha_{n-1} \wedge \alpha_n) \to (\Box\alpha_{n-1} \vee \Box\alpha_n))].$$

Now this is equivalent by propositional logic to

$$(\Diamond\bot \wedge \Box\gamma \wedge \Box\delta \wedge \neg\Box(\gamma \vee \delta)) \to ((\Box(\alpha_{n-1} \wedge \alpha_n) \to \Box\alpha_{n-1} \vee \Box\alpha_n)).$$

It follows from this and the above that

$$\vdash (\Diamond\bot \wedge \Box\gamma \wedge \Box\delta \wedge \neg\Box(\gamma \vee \delta) \wedge \Box\beta) \to (\Box\alpha_1 \vee \ldots \vee (\Box\alpha_{n-1} \vee \Box\alpha_n)),$$

that is,

$$\vdash (\Diamond\bot \wedge \Box\gamma \wedge \Box\delta \wedge \neg\Box(\gamma \vee \delta) \wedge \Box\beta) \to (\Box\alpha_1 \vee \ldots \vee \Box\alpha_n),$$

which is what we intended to prove.

(ii) For RcP_n, the proof is analogous, using nNP. ■

We need now to show that **nocp**, as axiomatized above, is indeed determined by the class of nocp-frames. We will do that later; first, we will present the remaining six Sylvan logics.

4 Prenormal Sylvan logics

Prenormal modal logics were introduced in by B. Chellas and K. Segerberg in [1]: a classical modal logic is *prenormal* if it provides the schema nK. The smallest prenormal logic was called **P** by Chellas and Segerberg. In their paper, they considered six prenormal logics: **P**, **PK**, and **PX**, and their extensions with the schema T ($\Box\alpha \to \alpha$). Using a more standard naming procedure, they could also be named **EnK**, **EK**, and **EX**. Chellas and Segerberg also introduced a family $\mathbf{PX}_1, \ldots, \mathbf{PX}_\omega$ of logics called *Cresswell logics*: these are prenormal logics providing the rule

(RC_n) $\alpha_1 \vee \ldots \vee \alpha_n$ / $(\Box\alpha_1 \wedge \ldots \wedge \Box\alpha_n) \to \Box\top$, for $n \geq 1$.

Semantically, Chellas and Segerberg use mixed frames $\langle U, N, Q, R, S\rangle$, where N is a set of normal points, Q a set of queer ones, and R and S are, respectively, an accessibility relation and a neighborhood function (with the proviso that $U \notin S(x)$, for every $x \in U$). Truth conditions for modalized formulas are the standard relational ones for $x \in N$, and the standard neigborhood semantics ones for $x \in Q$, that is:

- for $x \in Q$, $x \Vdash \Box\alpha$ iff $\{y \in U : y \Vdash \alpha\} \in S(x)$.

Notice that **nocp** itself is not a prenormal logic: nK is not valid in the class of all frames. Could it, however, be contained in **EnK**? We can easily show that this is not the case. Take this instance of **nocp**'s axiom qW: $\Diamond\bot \to (\Box(p \vee q) \to (\Box p \wedge \Box q))$. Now consider an **EnK** model $\langle U, N, Q, R, S, V\rangle$ such that $U = \{1, 2, 3\}$, $N = \{1\}$, $Q = \{2, 3\}$, $R = \emptyset$ and S is such that $S(2) = S(3) = \{\{1, 2\}\}$. Let us take $V(p) = \{1\}$ and $V(q) = \{2\}$. Notice that $U \notin S(x)$ for every $x \in Q$, so it is an **EnK** model according to [1]. Now we have: $2 \Vdash \neg\Box\top$, since $U \notin S(2)$, that is, $2 \Vdash \Diamond\bot$. And since $\|p \vee q\| = \{1, 2\}$ is in $S(2)$, $2 \Vdash \Box(p \vee q)$. However $\{2\} \notin S(2)$, for instance, so $2 \nVdash \Box q$ and thus $2 \nVdash \Box p \wedge \Box q$. But then qW is not **EnK**-valid. **EnK** and **nocp** are indeed different logics.

In this section we will consider extensions of **nocp** which are prenormal. They are the logics of frames which do not have perverse worlds, since nK fails to hold in such worlds. We thus have three logics: **noc**, **nc**, and **no**. To illustrate the relations among all logics considered in this paper, see Figure 2. This diagram also contains two extensions of **E** which are not prenormal (and were not considered in [1]), but are included in **EnK**: **EnC** and **EnM**. It is not difficult to show that semantics for these logics are pure neighborhood semantics with conditions (nc) and (nm) below (what we could also do alternatively for **EnK** with (nk)):

(nc) if $U \in S(x)$, $X \in S(x)$ and $Y \in S(x)$, then $X \cap Y \in S(x)$,

(nm) if $U \in S(x)$ and $X \cap Y \in S(x)$, then $X \in S(x)$ and $Y \in S(x)$,

(nk) if $U \in S(x)$, $X \in S(x)$ and $-X \cup Y \in S(x)$, then $Y \in S(x)$.

4.1 noc

Here we consider frames with only normal, opposite and contrary points, for short, *noc-frames*: these are structures $\langle U, N, O, C, P, R\rangle$ in which $P = \emptyset$, or, to simplify, just structures $\langle U, N, O, C, R\rangle$. The logic of the class of all noc-frames will be called **noc**.

To axiomatize **noc**, we could only drop the corresponding clause to perverse points on axiom nNP, obtaining

$$\Box\top \to ((\Box\alpha \wedge \Box\beta) \to \Box(\alpha \wedge \beta)).$$

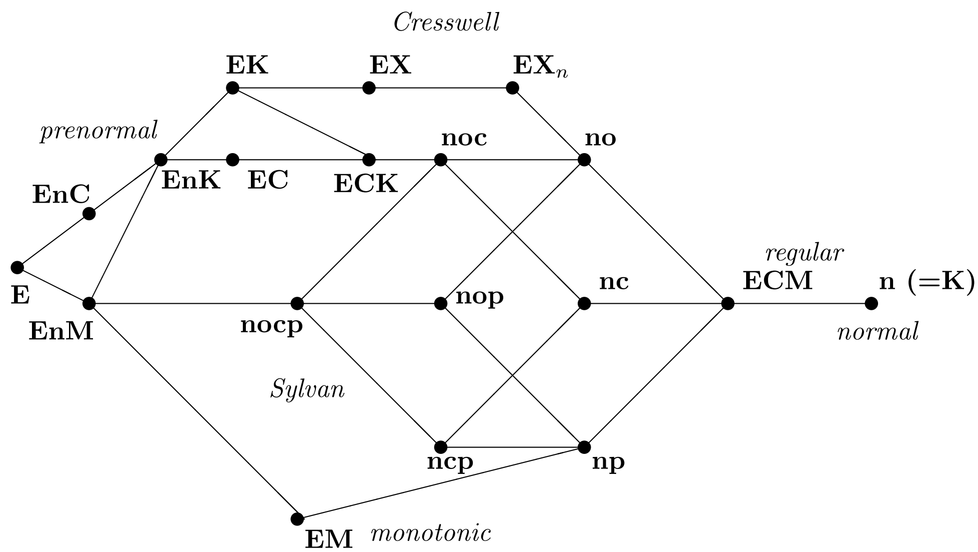

Figure 2: **E** and some of its extensions

But this, of course, is the schema nC. Together with nM, we have nF, and Lemma 3.2 of [1] tells us that a classical logic has nF iff it has nK iff it has nX iff it has RnK. As we can show from Table 1, K is valid in all non perverse points — and if a logic has K, it obviously has nK. So we could simply axiomatize **noc** adding the following schemes to **E**:

$$\text{(K)} \quad \Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta),$$

$$\text{(qW)} \quad \Diamond\bot \to (\Box(\alpha \vee \beta) \to (\Box\alpha \wedge \Box\beta)),$$

$$\text{(qOC)} \quad \Diamond\bot \to [((\Box\alpha \wedge \Box\beta) \to \Box(\alpha \vee \beta)) \vee (\Box(\gamma \wedge \delta) \to (\Box\gamma \vee \Box\delta))].$$

Obviously **noc** is a *prenormal* logic, because nK follows from K. The following two propositions give us examples of what hold in **noc**, and what not.

Proposition 4.1. **noc** *provides the following schemas and rules of inference:*

$$\text{(C)} \quad (\Box\alpha \wedge \Box\beta) \to \Box(\alpha \wedge \beta),$$

$$\text{(nF)} \quad \Box\top \to (\Box(\alpha \wedge \beta) \leftrightarrow (\Box\alpha \wedge \Box\beta)),$$

$$\text{(RnK)} \quad (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta \;/\; \Box\top \to ((\Box\alpha_1 \wedge \ldots \wedge \Box\alpha_n) \to \Box\beta), \textit{ for } n \geq 0,$$

Having C and K, **noc** also contains the classical logic **ECK**. (This inclusion is proper, as shown in Lemma 4.3.) It is, however, neither monotonic nor a Cresswell logic, as the next lemma shows.

Proposition 4.2. *The following schemas and rules are* not *provided in* **noc***:*

(M) $\Box(\alpha \wedge \beta) \to (\Box\alpha \wedge \Box\beta)$,

(X) $(\Box(\alpha \to \beta) \wedge (\Box(\beta \to \gamma)) \to \Box(\alpha \to \gamma)$,

(RM) $\alpha \to \beta \mathbin{/} \Box\alpha \to \Box\beta$,

(RC_n) $\alpha_1 \vee \ldots \vee \alpha_n \mathbin{/} (\Box\alpha_1 \wedge \ldots \wedge \Box\alpha_n) \to \Box\top$, *for* $n \geq 1$.

As we mentioned, RC_n is the rule typical of Cresswell logics (see [1]). Thus **noc**, in spite of being prenormal, is not a Cresswell logic. To wit, RC_n fails at contrary worlds. Since $p \vee \neg p$ is a tautology, it is a thesis of every Cresswell logic and, by RC_n, so is $(\Box p \wedge \Box\neg p) \to \Box\top$. But this is not valid in **noc**: one can easily see that this formula is falsifiable in contrary points. On the other hand, C is a thesis of **noc**, but not of $\mathbf{EX}_n$.

Proposition 4.3. **ECK** *is properly contained in* **noc***.*

Proof. Consider the formula $\neg(p \to q) \to (q \to p)$. It is a tautology, so it is a theorem of both **ECK** and **noc**. Now, since **noc** provides RqW, it follows that

$$\Diamond\bot \to (\Box(q \to p) \to \Box\neg(p \to q)) \tag{1}$$

is a thesis. However, it is not valid in **ECK**. Let $\mathfrak{M} = \langle U, S, V\rangle$ be a neighborhood model, where $U = \{1, 2, 3\}$, $V(p) = \{1, 2\}$, $V(q) = \{1, 3\}$, and S is such that

$$S(1) = S(2) = S(3) = \{\{1, 2\}\}\,.$$

This model satisfies the following conditions (where $x \in U$ and X and Y are any subsets of U):

(c) if $X \in S(x)$ and $Y \in S(x)$, then $X \cap Y \in S(x)$,

(k) if $X \in S(x)$ and $-X \cup Y \in S(x)$, then $Y \in S(x)$.

Thus, it is an ECK-model. We have $\|p \to q\| = \{1, 3\}$, $\|q \to p\| = \{1, 2\}$, and $-\|p \to q\| = \{3\}$. Now, since $\|q \to p\| \in S(1)$, $1 \Vdash \Box(q \to p)$, and since $-\|p \to q\| \notin S(1)$, $1 \nVdash \Box\neg(p \to q)$. Thus $1 \nVdash \Box(q \to p) \to \Box\neg(p \to q)$. However, $U \notin S(1)$, thus $1 \nVdash \Box\top$ and then $1 \Vdash \Diamond\bot$. Thus (1) fails in 1 and is not **ECK**-valid. ■

4.2 no

Here we consider frames with only normal and opposite worlds, for short, *no-frames*. We can drop the condition for contrary worlds on axiom OC, obtaining

$$\Diamond\bot \to (\Box\alpha \wedge \Box\beta \to \Box(\alpha \vee \beta)).$$

But this is the other direction of the consequent in qW. This way, we can axiomatize **no** by adding the following axioms to **E**:

(O) $\Box\top \vee \Box\bot$,

(K) $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$,

(qZ) $\Diamond\bot \to (\Box(\alpha \vee \beta) \leftrightarrow (\Box\alpha \wedge \Box\beta))$.

Naturally, **noc** is properly contained in **no**, since, for instance, X fails at contrary worlds, but holds in normal and opposite. **no** is thus a Cresswell logic, because it has RC_n, as is easily shown.

Proposition 4.4. **no** *also provides the following schemas and rules of inference:*

(O′) $\Diamond\bot \to \Box\bot$,

(RO_n) $\beta \to (\alpha_1 \vee \ldots \vee \alpha_n)\ /\ \Diamond\bot \to ((\Box\alpha_1 \wedge \ldots \wedge \Box\alpha_n) \to \Box\beta)$, *for* $n \geq 1$.

(RC_n) $\alpha_1 \vee \ldots \vee \alpha_n\ /\ (\Box\alpha_1 \wedge \ldots \wedge \Box\alpha_n) \to \Box\top$, *for* $n \geq 1$.

Proof. O′ follows easily from O. For RO_n, suppose $\vdash \beta \to (\alpha_1 \vee \ldots \vee \alpha_n)$. By RqW, we have $\vdash \Diamond\bot \to (\Box(\alpha_1 \vee \ldots \vee \alpha_n) \to \Box\beta)$, which gives us (*) $\vdash \Box(\alpha_1 \vee \ldots \vee \alpha_n) \to (\Diamond\bot \to \Box\beta)$. Now from axiom qZ we can easily obtain

$$\vdash \Diamond\bot \to ((\Box\alpha_1 \wedge \ldots \wedge \Box\alpha_n) \to \Box(\alpha_1 \vee \ldots \vee \alpha_n)).$$

From this and (*) we have

$$\vdash \Diamond\bot \to ((\Box\alpha_1 \wedge \ldots \wedge \Box\alpha_n) \to (\Diamond\bot \to \Box\beta)),$$

from which it follows, by classical propositional logic, that

$$\vdash \Diamond\bot \to ((\Box\alpha_1 \wedge \ldots \wedge \Box\alpha_n) \to \Box\beta).$$

For RC_n, suppose $\alpha_1 \vee \ldots \vee \alpha_n$ is a theorem. Then so is $\top \to (\alpha_1 \vee \ldots \vee \alpha_n)$. By RO_n, we have $\Diamond\bot \to ((\Box\alpha_1 \wedge \ldots \wedge \Box\alpha_n) \to \Box\top)$, from what it follows, by propositional reasoning, $(\Box\alpha_1 \wedge \ldots \wedge \Box\alpha_n) \to (\neg\Box\top \to \Box\top)$. Now, since $(\neg\Box\top \to \Box\top) \to \Box\top$ is a tautology, we have $(\Box\alpha_1 \wedge \ldots \wedge \Box\alpha_n) \to \Box\top$, which is the desired result. ■

4.3 nc

Here we consider frames with only normal and contrary points, for short, *nc-frames*. Dropping the condition for opposite points in OC gives us qY, so we can axiomatize **nc** by adding the following axioms to **E**:

(K) $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$,

(qY) $\Diamond\bot \to (\Box(\alpha \wedge \beta) \to (\Box\alpha \vee \Box\beta))$.

Proposition 4.5. **nc** *also provides the following schemas:*

(C) $(\Box\alpha \wedge \Box\beta) \to \Box(\alpha \wedge \beta)$,

(nF) $\Box\top \to (\Box(\alpha \wedge \beta) \leftrightarrow (\Box\alpha \wedge \Box\beta))$.

In spite of extending **noc**, **nc** is neither monotonic, nor regular, nor a Cresswell logic, as the following result shows.

Proposition 4.6. *The following schemas are* not *provided in* **nc**:

(M) $\Box(\alpha \wedge \beta) \to (\Box\alpha \wedge \Box\beta)$,

(F) $\Box(\alpha \wedge \beta) \leftrightarrow (\Box\alpha \wedge \Box\beta)$,

(X) $(\Box(\alpha \to \beta) \wedge (\Box(\beta \to \gamma)) \to \Box(\alpha \to \gamma)$,

(O) $\Box\bot \vee \Box\top$.

noc is properly included in **nc**: Y hols in normal and contrary points, but not in opposite ones. And since $\Box\bot \vee \Box\top$ is a thesis of **no**, but not of **nc**, neither logic contains the other.

5 Logics of frames with perverse points

In this section we present the remaining three Sylvan logics, those whose frames contain perverse points, that is, **nop**, **ncp**, and **np**.

5.1 nop

An axiomatics is provided by adding to **E** the following schemes:

(nM) $\Box\top \to (\Box(\alpha \wedge \beta) \to (\Box\alpha \wedge \Box\beta))$,

(nNP) $\Box\top \to [((\Box\alpha \wedge \Box\beta) \to \Box(\alpha \wedge \beta)) \vee (\Box(\gamma \vee \delta) \to (\Box\gamma \vee \Box\delta))]$,

(qZ) $\Diamond\bot \to (\Box(\alpha \vee \beta) \leftrightarrow (\Box\alpha \wedge \Box\beta))$.

nop properly contains **nocp**, since V is a thesis, but it fails in points which are contrary. **nop** is also properly contained both in **no** and **np**: $\Box\bot \vee \Box\top$ is a thesis of **no** but not of **nop**; M holds in **np** but not in **nop**.

5.2 ncp

For the axiomatics, just add to **E** the following:

(nM) $\Box\top \to (\Box(\alpha \wedge \beta) \to (\Box\alpha \wedge \Box\beta))$,

(nNP) $\Box\top \to [((\Box\alpha \wedge \Box\beta) \to \Box(\alpha \wedge \beta)) \vee (\Box(\gamma \vee \delta) \to (\Box\gamma \vee \Box\delta))]$,

(qW) $\Diamond\bot \to (\Box(\alpha \vee \beta) \to (\Box\alpha \wedge \Box\beta))$,

(qY) $\Diamond\bot \to (\Box(\alpha \wedge \beta) \to (\Box\alpha \vee \Box\beta))$.

ncp properly contains **nocp**, since Y is a thesis, but it fails in opposite points. **ncp** is also properly contained both in **nc** and **np**: C is a thesis of **nc** but not of **ncp**; M holds in **np** but not in **ncp**.

5.3 np

For the axiomatics, just add to **E** the following:

(M) $\Box(\alpha \wedge \beta) \to (\Box\alpha \wedge \Box\beta)$,

(nNP) $\Box\top \to [((\Box\alpha \wedge \Box\beta) \to \Box(\alpha \wedge \beta)) \vee (\Box(\gamma \vee \delta) \to (\Box\gamma \vee \Box\delta))]$.

Notice that **np**, having M, extends the smallest monotonic logic **EM**. It is, however, neither normal nor regular, since C and K fail at perverse points. But it does extend **EM**, as the next proposition shows:

Proposition 5.1. **np** *is a proper extension of* **EM**.

Proof. Consider the formula $(p \to q) \vee \neg q$. It is a tautology, so it is a theorem of both **EM** and **np**. Now, since **np** has RcP_n, for every n, it follows that (taking $k = 2$ and letting β_1 and γ be the same formula p)

$$(\Box\top \wedge \Box p \wedge \Box r \wedge \neg\Box(p \wedge r)) \to (\Box q \vee \Box\neg q) \tag{2}$$

is a thesis. However, it is not valid in **EM**. Let $\mathfrak{M} = \langle U, S, V\rangle$ be a neighbourhood model, where $U = \{1, 2, 3, 4\}$, $V(p) = \{1, 2\}$, $V(q) = \{1, 3\}$, $V(r) = \{1, 4\}$, and S is such that

$$S(1) = S(2) = S(3) = S(4) = \big\{\{1, 2\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, U\big\}.$$

It is easy to see that this model satisfies the following condition (where $x \in U$ and X and Y are any subsets of U):

(m) if $X \cap Y \in S(x)$ then $X \in S(x)$ and $Y \in S(x)$.

So it is an **EM**-model. We have $\|p\| = \{1, 2\}$, $\|q\| = \{1, 3\}$, $\|r\| = \{1, 4\}$, $\|\neg q\| = \{2, 4\}$, and $\|p \wedge r\| = \{1\}$. Now, since $\|p\| \in S(1)$ and $\|r\| \in S(1)$, $1 \Vdash \Box p$ and $1 \Vdash \Box r$. On the other hand, $\|p \wedge r\| \notin S(1)$, $1 \nVdash \Box(p \wedge r)$, so $1 \Vdash \neg\Box(p \wedge r)$. And since $U \in S(1)$, $1 \Vdash \Box\top$. Thus the antecedent of (2) is true at world 1. However, neither $\|q\|$ nor $\|\neg q\|$ are in $S(1)$, so $1 \nVdash \Box q$ and $1 \nVdash \Box\neg q$. Hence, the consequent of (2), and thus (2) itself, is false in world 1, so it is not **EM**-valid. ■

6 Completeness

Completeness for all Sylvan logics discussed here can be proved in the standard way using canonical models. Where Λ is a logic, let $\mathfrak{S}_\Lambda$ be the set of all maximal consistent sets (MCSs) of formulas in Λ.

Definition 6.1. *A set Γ of formulas is Λ-inconsistent if there is a finite subset of formulas $\{\alpha_1, \ldots, \alpha_n\} \subseteq \Gamma$ such that $\vdash_\Lambda \neg(\alpha_1 \wedge \ldots \wedge \alpha_n)$; otherwise Γ is Λ-consistent.*

If Γ is finite, i.e., $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$, Γ is Λ-consistent if and only if $\nvdash_\Lambda \neg(\gamma_1 \wedge \ldots \wedge \gamma_n)$. And a singleton $\{\alpha\}$ is Λ-consistent, of course, if and only if $\nvdash_\Lambda \neg\alpha$.

Definition 6.2. *Let Γ be a set of formulas and α a formula. We say that $\Gamma \vdash_\Lambda \alpha$ if there is a finite subset of formulas $\{\alpha_1, \ldots, \alpha_n\} \subseteq \Gamma$ such that $\vdash_\Lambda (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \alpha$.*

Definition 6.3. *A set Γ of formulas is* maximal *if, for every formula α, either $\alpha \in \Gamma$ or $\neg\alpha \in \Gamma$. Γ is a* maximal consistent set *(MCS) if it is maximal and consistent.*

The proofs of the following lemmas are standard, so we will omit them.

Lemma 6.4. *Let Δ be an MCS, and α and β any formulas. Then:*

(i) $\bot \notin \Delta$

(ii) *$\alpha \in \Delta$ iff $\neg\alpha \notin \Delta$;*

(iii) *$\alpha \wedge \beta \in \Delta$ iff $\alpha \in \Delta$ and $\beta \in \Delta$;*

(iv) *$\alpha \vee \beta \in \Delta$ iff $\alpha \in \Delta$ or $\beta \in \Delta$;*

(v) *$\alpha \to \beta \in \Delta$ iff $\alpha \notin \Delta$ or $\beta \in \Delta$;*

(vi) *$\alpha \leftrightarrow \beta \in \Delta$ iff $\alpha \in \Delta$ and $\beta \in \Delta$, or $\alpha \notin \Delta$ and $\beta \notin \Delta$;*

(vii) *if $\alpha \in \Delta$ and $\alpha \to \beta \in \Delta$ then $\beta \in \Delta$.*

Lemma 6.5 (Lindenbaum)**.** *Let Γ be a consistent set of formulas. Then there is an MCS Δ such that $\Gamma \subseteq \Delta$.*

Lemma 6.6. *$\vdash_\Lambda \alpha$ iff for every Λ-MCS Γ, $\alpha \in \Gamma$.*

We need now a few more concepts, which will be useful for several logics.

Definition 6.7. *Let Γ be a Λ-consistent set of formulas. We say that:*

(i) Γ *is* C-distributive *if there are no formulas α and β such that $\{\Box\alpha, \Box\beta\} \subseteq \Gamma$ and $\Gamma \vdash_\Lambda \neg\Box(\alpha \wedge \beta)$;*

(ii) Γ *is* D-distributive *if there are no formulas α and β such that $\{\Box\alpha, \Box\beta\} \subseteq \Gamma$ and $\Gamma \vdash_\Lambda \neg\Box(\alpha \vee \beta)$.*

Lemma 6.8. *If Γ is a C-distributive Λ-MCS and $\{\Box\alpha_1, \ldots, \Box\alpha_n\} \subseteq \Gamma$, for $n > 0$, then $\Gamma \vdash_\Lambda \Box(\alpha_1 \wedge \ldots \wedge \alpha_n)$.*

Proof. Start with $\Box\alpha_1$ and $\Box\alpha_2$. Since Γ is C-distributive, $\Gamma \nvdash_\Lambda \neg\Box(\alpha_1 \wedge \alpha_2)$ and, being an MCS, $\Gamma \vdash_\Lambda \Box(\alpha_1 \wedge \alpha_2)$. Take now $\Box(\alpha_1 \wedge \alpha_2)$ and $\Box\alpha_3$ to obtain $\Gamma \vdash_\Lambda \Box(\alpha_1 \wedge \alpha_2 \wedge \alpha_3)$. Repeating this construction will give us $\Gamma \vdash_\Lambda \Box(\alpha_1 \wedge \ldots \wedge \alpha_n)$. ■

Lemma 6.9. *If Γ is an D-distributive Λ-MCS and $\{\Box\alpha_1, \ldots, \Box\alpha_n\} \subseteq \Gamma$, for $n > 0$, then $\Gamma \vdash_\Lambda \Box(\alpha_1 \vee \ldots \vee \alpha_n)$.*

Proof. Analogous to the preceding lemma. ■

6.1 Completeness for nocp

We will be using canonical models, in which the points are MCSs. Since we have normal and non-normal points, we need to specify when a certain MCS is normal, or opposite, and so on.

Definition 6.10. *Let Δ be a maximal consistent set.*

(a) Δ *is* normal *if $\Box\top \in \Delta$ and either (i) for every α, $\Box\alpha \in \Delta$ or (ii) for at least some α, $\neg\Box\alpha \in \Delta$ and Δ is C-distributive.*

(b) Δ *is* perverse *if $\Box\top \in \Delta$ and Δ is not C-distributive.*

(c) Δ *is* contrary *if $\Diamond\bot \in \Delta$ and either (i) for every α, $\neg\Box\alpha \in \Delta$ or (ii) Δ is not D-distributive.*

(d) Δ *is* opposite *if $\Diamond\bot \in \Delta$, for at least some α, $\Box\alpha \in \Delta$ and Δ is D-distributive.*

The intuition behind the above definition is this. Since R can be any relation, a model may contain points which do not have any accessible points. If x is such a point, and x is either normal or opposite, then $x \Vdash \Box\alpha$ for every α. Let $T(x)$ be the set of all formulas true at x. $T(x)$ is obviously an MCS, but when it comes to classify it as normal or opposite, in order to build a canonical model, we could do it either way. So we will choose to call it normal. And should x be contrary or perverse, then $x \Vdash \neg\Box\alpha$ for every α. Accordingly, we will choose to call the corresponding MCS, $T(x)$, contrary.

Barring this case, x will have at least one accessible point. If x is normal or perverse, then $\Box\top$ will be true at x ($\top$ is true at every point), and also $\neg\Box\alpha$ for at least one α — for instance, $\neg\Box\bot$ ($\bot$ is false at every point). Only this, however, would not distinguish normal from perverse points — we need the stronger requirement of C-distributivity (C holds in normal points, but fails in perverse ones).

Similarly, the requirement of D-distributivity will be used to distinguish opposite (D-distributive) and contrary (non D-distributive) points. We can then show that the above definition exhaust all possibilities regarding MCSs.

Proposition 6.11. *Every MCS is either normal, or opposite, or contrary, or perverse.*

Proof. Let Δ be an MCS. Suppose, first, that for every α, $\Box\alpha \in \Delta$. By 6.10.(i), Δ is normal. If, on the other hand, $\neg\Box\alpha \in \Delta$, for every α, then by 6.10.(iii), Δ is contrary.

Suppose, then, that for some α and some β, $\{\Box\alpha, \neg\Box\beta\} \subseteq \Delta$. Now either $\Box\top \in \Delta$ or $\Diamond\bot \in \Delta$. Suppose $\Box\top \in \Delta$. If Δ is C-distributive, it is normal; if not, it is perverse. On the other hand, if $\Diamond\bot \in \Delta$, then Δ is opposite if D-distributive, and contrary if not. ■

Definition 6.12. *Let Γ and Δ be MCSs. Then:*

(a) $n(\Gamma) = \{\alpha \mid \Box\alpha \in \Gamma\}$*;*

(b) $n^-(\Gamma) = \{\neg\alpha \mid \Box\alpha \in \Gamma\}$*;*

(c) $p(\Gamma) = \{\neg\alpha \mid \neg\Box\alpha \in \Gamma\}$*;*

(d) $p^-(\Gamma) = \{\alpha \mid \neg\Box\alpha \in \Gamma\}$*;*

(e) $\Gamma\rho\Delta$ *iff* Γ *is normal and* $n(\Gamma) \subseteq \Delta$*, or*
Γ *is opposite and* $n^-(\Gamma) \subseteq \Delta$*, or*
Γ *is contrary and* $p^-(\Gamma) \subseteq \Delta$*, or*
Γ *is perverse and* $p(\Gamma) \subseteq \Delta$*.*

We have now four important lemmas.

Lemma 6.13. *Let Γ be a C-distributive MCS containing $\Box\top$ and $\neg\Box\alpha$, for some formula α. Then $n(\Gamma) \cup \{\neg\alpha\}$ is consistent.*

Proof. Suppose that $n(\Gamma) \cup \{\neg\alpha\}$ is inconsistent. Since $\Box\top \in \Gamma$, $n(\Gamma) \neq \emptyset$. Now, if $n(\Gamma) \cup \{\neg\alpha\}$ is inconsistent, then, for some finite subset $\{\beta_1, \ldots, \beta_n\}$ of $n(\Gamma)$, we have $\vdash \neg(\beta_1 \wedge \ldots \wedge \beta_n \wedge \neg\alpha)$. (Should the inconsistency be in $n(\Gamma)$, we would have $\vdash \neg(\beta_1 \wedge \ldots \wedge \beta_n)$, but from this the formula above follows, adding $\neg\alpha$.) It thus follows that $\vdash \beta_1 \wedge \ldots \wedge \beta_n \to \alpha$.

Recall that RnM is provided in **nocp**. We then have $\vdash \Box\top \to (\Box(\beta_1 \wedge \ldots \wedge \beta_n) \to \Box\alpha)$, from which it tautologically follows that $\vdash \Box\top \to (\neg\Box\alpha \to \neg\Box(\beta_1 \wedge \ldots \wedge \beta_n))$.

Since $\Box\top$ and $\neg\Box\alpha \in \Gamma$, it follows that $\Gamma \vdash \neg\Box(\beta_1 \wedge \ldots \wedge \beta_n)$. On the other hand, since $\{\beta_1, \ldots, \beta_n\} \subseteq n(\Gamma)$, $\{\Box\beta_1, \ldots, \Box\beta_n\} \subseteq \Gamma$, and it follows from Lemma 6.8 that $\Gamma \vdash \Box(\beta_1 \wedge \ldots \wedge \beta_n)$, what makes Γ inconsistent, against the hypothesis of the lemma. Thus, $n(\Gamma) \cup \{\neg\alpha\}$ is consistent. ■

Lemma 6.14. *Let Γ be an D-distributive MCS such that $\Diamond\bot \in \Gamma$, $\neg\Box\alpha \in \Gamma$ and, for at least some formula β, $\Box\beta \in \Gamma$. Then $n^-(\Gamma) \cup \{\alpha\}$ is consistent.*

Proof. Since there is at least one $\Box\beta \in \Gamma$, $n^-(\Gamma) \neq \emptyset$. So suppose $n^-(\Gamma) \cup \{\alpha\}$ is not consistent. Then, for some $\beta_1, \ldots, \beta_n$ such that $\Box\beta_i \in \Gamma$, $\vdash \neg(\neg\beta_1 \wedge \ldots \wedge \neg\beta_n \wedge \alpha)$. From this it follows that $\vdash \alpha \to (\beta_1 \vee \ldots \vee \beta_n)$ and, by RqW, that $\vdash \Diamond\bot \to (\Box(\beta_1 \vee \ldots \vee \beta_n) \to \Box\alpha$. Now, since $\{\Box\beta_1, \ldots, \Box\beta_n\} \subseteq \Gamma$ and Γ is D-distributive, $\Box(\beta_1 \vee \ldots \vee \beta_n) \in \Gamma$ by Lemma 6.9. Since we also have $\Diamond\bot \in \Gamma$, $\Gamma \vdash \Box\alpha$, what makes Γ inconsistent against the hypothesis. ■

Lemma 6.15. *Let Γ be a consistent, but not D-distributive MCS such that $\Diamond\bot \in \Gamma$ and $\Box\alpha \in \Gamma$. Then $p^-(\Gamma) \cup \{\neg\alpha\}$ is consistent.*

Proof. Since Γ is not D-distributive, there are formulas γ and δ such that $\{\Box\gamma, \Box\delta\} \subseteq \Gamma$ and $\Gamma \vdash \neg\Box(\gamma \vee \delta)$, so $\neg\Box(\gamma \vee \delta) \in \Gamma$ and $p^-(\Gamma) \neq \emptyset$. Suppose now $p^-(\Gamma) \cup \{\neg\alpha\}$ is inconsistent. Thus, for some $\beta_1, \ldots, \beta_n$ such that $\neg\Box\beta_i \in \Gamma$, $\vdash \neg(\beta_1 \wedge \ldots \wedge \beta_n \wedge \neg\alpha)$. But from this it follows that $\vdash (\beta_1 \wedge \ldots \wedge \beta_n) \to \alpha$. Since RvC_n is provided by **nocp**, we have

$$\vdash (\Diamond\bot \wedge \Box\gamma \wedge \Box\delta \wedge \neg\Box(\gamma \vee \delta) \wedge \Box\alpha) \to (\Box\beta_1 \vee \ldots \vee \Box\beta_n),$$

from which it follows that

$$\vdash (\Diamond\bot \wedge \Box\gamma \wedge \Box\delta \wedge \neg\Box(\gamma \vee \delta) \wedge \neg\Box\beta_1 \wedge \ldots \wedge \neg\Box\beta_n) \to \neg\Box\alpha.$$

However, $\{\Diamond\bot, \Box\gamma, \Box\delta, \neg\Box(\gamma \vee \delta), \neg\Box\beta_1, \ldots, \neg\Box\beta_n\} \subseteq \Gamma$, so $\Gamma \vdash \neg\Box\alpha$, what makes Γ inconsistent against the hypothesis. Thus, $p^-(\Gamma) \cup \{\alpha\}$ is consistent. ■

Lemma 6.16. *Let Γ be a not C-distributive MCS such that $\Box\top \in \Gamma$ and $\Box\alpha \in \Gamma$. Then $p(\Gamma) \cup \{\alpha\}$ is consistent*

Proof. Since Γ is not C-distributive, there are formulas γ and δ such that $\{\Box\gamma, \Box\delta\} \subseteq \Gamma$ and $\Gamma \vdash \neg\Box(\gamma \wedge \delta)$, so $\neg\Box(\gamma \wedge \delta) \in \Gamma$ and $p(\Gamma) \neq \emptyset$. Suppose now $p(\Gamma) \cup \{\alpha\}$ is inconsistent. Thus, for some finite subset $\neg\Box\beta_1, \ldots, \neg\Box\beta_n$ of Γ, we have that $\{\neg\beta_1, \ldots, \neg\beta_n, \alpha\}$ is inconsistent, that is, $\vdash \neg(\neg\beta_1 \wedge \ldots \wedge \neg\beta_n \wedge \alpha)$. From this it follows that $\vdash \alpha \to \beta_1 \vee \ldots \vee \beta_n$. Since RcP_n is provided by **nocp**, we have

$$\vdash (\Box\top \wedge \Box\gamma \wedge \Box\delta \wedge \neg\Box(\gamma \wedge \delta) \wedge \Box\alpha) \to (\Box\beta_1 \vee \ldots \vee \Box\beta_n),$$

from which it follows that

$$\vdash (\Box\top \wedge \Box\gamma \wedge \Box\delta \wedge \neg\Box(\gamma \wedge \delta) \wedge \neg\Box\beta_1 \wedge \ldots \wedge \neg\Box\beta_n) \to \neg\Box\alpha.$$

However, $\{\Box\top, \Box\gamma, \Box\delta, \neg\Box(\gamma \wedge \delta), \neg\Box\beta_1, \ldots, \neg\Box\beta_n\} \subseteq \Gamma$, so $\Gamma \vdash \neg\Box\alpha$, what makes Γ inconsistent against the hypothesis. Thus, $p(\Gamma) \cup \{\alpha\}$ is consistent. ∎

We now define a canonical model for **nocp** and, by extension, for all other Sylvan logics — we only need to drop one or more sets of non-normal points. Where α is some formula, let $|\alpha|_\Lambda$ be the set of all Λ-MCSs Γ such that $\alpha \in \Gamma$.

Definition 6.17. *Let Λ be a Sylvan logic. We say that $\mathfrak{M}_\Lambda = \langle U_\Lambda, N_\Lambda, O_\Lambda, C_\Lambda, P_\Lambda, R_\Lambda, V_\Lambda\rangle$ is the canonical model for Λ iff it satisfies the following conditions:*

(i) $U_\Lambda = \mathfrak{S}_\Lambda$;

(ii) $N_\Lambda = \{\Gamma \in \mathfrak{S}_\Lambda : \Gamma \text{ is normal}\}$;

(iii) $O_\Lambda = \{\Gamma \in \mathfrak{S}_\Lambda : \Gamma \text{ is opposite}\}$;

(iv) $C_\Lambda = \{\Gamma \in \mathfrak{S}_\Lambda : \Gamma \text{ is contrary}\}$;

(v) $P_\Lambda = \{\Gamma \in \mathfrak{S}_\Lambda : \Gamma \text{ is perverse}\}$;

(vi) $R_\Lambda = \rho$;

(vii) $V_\Lambda(\mathbf{p}) = |\mathbf{p}|_\Lambda$, *for every* $\mathbf{p} \in \Phi$.

Lemma 6.18. *Let $\mathfrak{M}$ be a canonical model for a logic Λ. Then, for every wff α and every $\Gamma \in U$, $\mathfrak{M}, \Gamma \Vdash \alpha$ iff $\alpha \in \Gamma$.*

Proof. By induction on formulas. Let Γ be some element of U. We show only the modal case, so let $\alpha = \Box\beta$. We have four cases to consider, according to whether Γ is normal, opposite, contrary or perverse.

(i) Suppose that $\Gamma \in N_\Lambda$. By definition, $\Gamma \Vdash \Box\beta$ iff for every Δ such that $\Gamma\rho\Delta$, $\Delta \Vdash \beta$. Suppose $\Gamma \nVdash \Box\beta$. Then there is some Δ such that $\Gamma\rho\Delta$ and $\Delta \nVdash \beta$. By the induction hypothesis (henceforth IH), for every $\Delta \in U$ we have that $\Delta \Vdash \beta$ iff $\beta \in \Delta$. Thus, $\beta \notin \Delta$. Since $\Gamma\rho\Delta$, $n(\Gamma) \subseteq \Delta$ and it follows that $\Box\beta \notin \Gamma$.

On the other hand, suppose that $\Box\beta \notin \Gamma$. Then $\neg\Box\beta \in \Gamma$ and, since Γ is also normal, by definition Γ is C-distributive and $\Box\top \in \Gamma$; hence $n(\Gamma) \neq \emptyset$. From lemma 6.13 it follows that $n(\Gamma) \cup \{\neg\beta\}$ is consistent and, from Lindenbaum's lemma, that it is included in some MCS Δ. But then $\Gamma\rho\Delta$ and $\neg\beta \in \Delta$. It follows that $\beta \notin \Delta$ and, by IH, that $\Delta \nVdash \beta$. Thus $\Gamma \nVdash \Box\beta$.

(ii) Suppose that $\Gamma \in O_\Lambda$. By definition, $\Gamma \Vdash \Box\beta$ iff for every Δ such that $\Gamma\rho\Delta$, $\Delta \nVdash \beta$. Suppose $\Gamma \nVdash \Box\beta$. Then there is some Δ such that $\Gamma\rho\Delta$ and $\Delta \Vdash \beta$. By IH, for every $\Delta \in U$ we have that $\Delta \Vdash \beta$ iff $\beta \in \Delta$. Hence, $\beta \in \Delta$ and $\neg\beta \notin \Delta$. Since $\Gamma\rho\Delta$, $n^-(\Gamma) \subseteq \Delta$ and so $\Box\beta \notin \Gamma$.

On the other hand, suppose that $\Box\beta \notin \Gamma$. Then $\neg\Box\beta \in \Gamma$. Since Γ is opposite, by definition $\Diamond\bot \in \Gamma$, Γ is D-distributive and, for at least some γ, $\Box\gamma \in \Gamma$, so $n^-(\Gamma) \neq \emptyset$. From lemma 6.14 it follows that $n^-(\Gamma) \cup \beta$ is consistent and, from Lindenbaum's lemma, that it is included in some MCS Δ. But then $\Gamma\rho\Delta$ and $\beta \in \Delta$. It follows, by IH, that $\Delta \Vdash \beta$. Thus $\Gamma \nVdash \Box\beta$.

(iii) Suppose now that that $\Gamma \in C_\Lambda$. By definition, $\Gamma \Vdash \Box\beta$ iff there is some Δ such that $\Gamma\rho\Delta$ and $\Delta \nVdash \beta$. Suppose $\Gamma \Vdash \Box\beta$. Then there is some Δ such that $\Gamma\rho\Delta$ and $\Delta \nVdash \beta$. By IH, for every $\Delta \in U$ we have that $\Delta \Vdash \beta$ iff $\beta \in \Delta$. Thus, $\beta \notin \Delta$. Since $\Gamma\rho\Delta$, that is, $p^-(\Gamma) \subseteq \Delta$, if $\neg\Box\beta$ were in Γ we would have $\beta \in \Delta$, what cannot be. It follows that $\Box\beta \in \Gamma$.

On the other hand, suppose that $\Box\beta \in \Gamma$. Since Γ is a contrary MCS, by definition $\Diamond\bot \in \Gamma$ and, given that $\Box\beta \in \Gamma$, Δ is not D-distributive. Now this means that there are γ and δ, $\{\Box\gamma, \Box\delta\} \subseteq \Gamma$ and $\neg\Box(\gamma \vee \delta) \in \Gamma$, so $p^-(\Gamma) \neq \emptyset$. From lemma 6.15 it follows that $p^-(\Gamma) \cup \{\neg\beta\}$ is consistent and, from Lindenbaum's lemma, that it is included in some MCS Δ. But then $\Gamma\rho\Delta$ and $\neg\beta \in \Delta$. It follows by IH that $\Delta \nVdash \beta$. Thus there is some MCS Δ such that $\Gamma\rho\Delta$ and $\Delta \nVdash \beta$, that is, $\Gamma \Vdash \Box\beta$.

(iv) Suppose now that that $\Gamma \in P_\Lambda$. By definition, $\Gamma \Vdash \Box\beta$ iff there is some Δ such that $\Gamma\rho\Delta$ and $\Delta \Vdash \beta$. Suppose $\Gamma \Vdash \Box\beta$. Then there is some Δ such that $\Gamma\rho\Delta$ and $\Delta \Vdash \beta$. By IH, for every $\Delta \in U$ we have that $\Delta \Vdash \beta$ iff $\beta \in \Delta$. Thus, $\beta \in \Delta$. Since $\Gamma\rho\Delta$, that is, $p(\Gamma) \subseteq \Delta$, if $\neg\Box\beta$ were in Γ we would have $\neg\beta \in \Delta$, what cannot be. It follows that $\Box\beta \in \Gamma$.

On the other hand, suppose that $\Box\beta \in \Gamma$. Since Γ is a perverse MCS, $\Box\top \in \Gamma$ and Γ is not C-distributive. Now this means that there are γ and δ, $\{\Box\gamma, \Box\delta\} \subseteq \Gamma$ and $\neg\Box(\gamma \wedge \delta) \in \Gamma$, so $p(\Gamma) \neq \emptyset$. From lemma 6.16 it follows that $p(\Gamma) \cup \beta$ is consistent and, from Lindenbaum's lemma, is is included in some MCS Δ. But then $\Gamma\rho\Delta$ and $\beta \in \Delta$. It follows by IH that $\Delta \Vdash \beta$. Thus there is some MCS Δ such that $\Gamma\rho\Delta$ and $\Delta \Vdash \beta$, that is, $\Gamma \Vdash \Box\beta$.

It follows that $\Gamma \Vdash \Box\beta$ iff $\Box\beta \in \Gamma$. ■

6.2 Completeness for the remaining systems

The overall strategy is the same; we only need to adapt the definitions of normal, opposite, contrary and perverse MCSs to each of the logics considered and prove the corresponding versions of Lemmas 6.13–6.16.

6.2.1 Completeness for noc

Not having perverse points simplifies our definition of a normal MCS: an MCS Δ is said to be *normal* if $\Box\top \in \Delta$. As for contrary and opposite points, there are no changes with regard to Definition 6.10, and we can show without difficulty that every MCS is either normal, opposite or contrary.

For completeness, we have only a small change in the proof of the following lemma concerning normal MCSs.

Lemma 6.19. *Let Γ be an MCS containing $\Box\top$ and $\neg\Box\alpha$, for some formula α. Then $n(\Gamma) \cup \{\neg\alpha\}$ is consistent.*

Proof. Suppose that $n(\Gamma) \cup \{\neg\alpha\}$ is inconsistent. Since $\Box\top \in \Gamma$, $n(\Gamma) \neq \emptyset$. Now, if $n(\Gamma) \cup \{\neg\alpha\}$ is inconsistent, then, for some finite subset $\{\beta_1, \ldots, \beta_n\}$ of $n(\Gamma)$, we have $\vdash \neg(\beta_1 \wedge \ldots \wedge \beta_n \wedge \neg\alpha)$. It thus follows that: $\vdash (\beta_1 \wedge \ldots \wedge \beta_n) \to \alpha$, and, since RnK is provided in **noc**, we have $\vdash \Box\top \to ((\Box\beta_1 \wedge \ldots \wedge \Box\beta_n) \to \Box\alpha)$. Now, since $\{\beta_1, \ldots, \beta_n\} \subseteq n(\Gamma)$, $\{\Box\top, \Box\beta_1, \ldots, \Box\beta_n\} \subseteq \Gamma$, and it follows that $\Gamma \vdash \Box\alpha$, what makes Γ inconsistent, against the hypothesis. ■

6.2.2 Completeness for no

Definition 6.20. *An MCS Δ is said to be* normal *if $\Box\top \in \Delta$; otherwise, Δ is* opposite.

Notice that to show that every MCSs is either normal or opposite we need O, that is, $\Box\top \vee \Box\bot$. Without this thesis we could have an MCS Δ such that $\neg\Box\alpha \in \Delta$ for every α (a *contrary* **nocp**-MCS). Having O rules this out.

Lemma 6.21. *Let Γ be an MCS containing $\Box\top$ and $\neg\Box\alpha$, for some formula α. Then $n(\Gamma) \cup \{\neg\alpha\}$ is consistent.*

Proof. As in **noc**. ■

Lemma 6.22. *Let Γ be an MCS such that $\Diamond\bot \in \Gamma$ and, for some formula α, $\neg\Box\alpha \in \Gamma$. Then $n^-(\Gamma) \cup \{\alpha\}$ is consistent*

Proof. Since **no** has O, Γ also contains $\Box\bot$, and so $n^-(\Gamma) \neq \emptyset$. Suppose $n^-(\Gamma) \cup \{\alpha\}$ is inconsistent. Thus, for some finite subset $\Box\beta_1, \ldots, \Box\beta_n$ of Γ, we have $\vdash \neg(\neg\beta_1 \wedge \ldots \wedge \neg\beta_n \wedge \alpha)$, from what $\vdash \alpha \to \beta_1 \vee \ldots \vee \beta_n$ follows. Using RqW, we obtain

$$\vdash \Diamond\bot \to (\Box(\beta_1 \vee \ldots \vee \beta_n) \to \Box\alpha).$$

Since qZ is an axiom, we then have

$$\vdash \Diamond\bot \to ((\Box\beta_1 \wedge \ldots \wedge \Box\beta_n) \to \Box\alpha).$$

Since $\{\Diamond\bot, \Box\beta_1, \ldots, \Box\beta_n\} \subseteq \Gamma$, $\Gamma \vdash \Box\alpha$ and is not consistent, against the lemma's hypothesis. Thus, $n^-(\Gamma) \cup \{\alpha\}$ is consistent. ■

6.2.3 Completeness for nc

Definition 6.23. *A MCS Δ is said to be* normal *if $\Box\top \in \Delta$; otherwise, Δ is* contrary.

We have only a small change in the proof of the following lemma concerning contrary worlds.

Lemma 6.24. *Let Γ be an MCS such that $\Diamond\bot \in \Gamma$, $\Box\alpha \in \Gamma$. Then $p^{-}(\Gamma) \cup \{\neg\alpha\}$ is consistent.*

Proof. Given that $\Diamond\bot \in \Gamma$, $\neg\Box\top \in \Gamma$, so $p^{-}(\Gamma) \neq \emptyset$. Suppose $p^{-}(\Gamma) \cup \{\neg\alpha\}$ is not consistent. Then, for some $\neg\Box\beta_1, \ldots, \neg\Box\beta_n$ in Γ, we have $\vdash \neg(\beta_1 \wedge \ldots \wedge \beta_n \wedge \neg\alpha)$, from what $\vdash (\beta_1 \wedge \ldots \wedge \beta_n) \to \alpha$ follows. Using RqW we obtain $\vdash \Diamond\bot \to (\Box\alpha \to \Box(\beta_1 \wedge \ldots \wedge \beta_n))$, from what we get, using qY, which is also provided in **nc**, $\vdash \Diamond\bot \to (\Box\alpha \to (\Box\beta_1 \vee \ldots \vee \Box\beta_n))$, and thus $\vdash \Diamond\bot \to (\neg(\Box\beta_1 \vee \ldots \vee \Box\beta_n) \to \neg\Box\alpha)$. Now this is equivalent to $\vdash \Diamond\bot \to ((\neg\Box\beta_1 \wedge \ldots \wedge \neg\Box\beta_n) \to \neg\Box\alpha)$, and, since $\{\Diamond\bot, \neg\Box\beta_1, \ldots, \neg\Box\beta_n\} \subseteq \Gamma$, $\neg\Box\alpha \in \Gamma$, making Γ inconsistent against the hypothesis. ∎

6.2.4 Completeness for nop

Normal MCSs are as in Definition 6.10. As for perverse and opposite ones, we have:

Definition 6.25.

(a) *An MCS Δ is* perverse *if either (i) for every α, $\neg\Box\alpha \in \Delta$ or (ii) $\Box\top \in \Delta$ and Δ is not C-distributive.*

(b) *An MCS Δ is said to be* opposite *if $\Diamond\bot \in \Delta$ and, for at least one formula α, $\Box\alpha \in \Delta$.*

We have only a small change in the proofs of the lemmas concerning opposite and perverse MCSs.

Lemma 6.26. *Let Γ be an MCS such that $\Diamond\bot \in \Gamma$, $\neg\Box\alpha \in \Gamma$ and, for at least some formula β, $\Box\beta \in \Gamma$. Then $n^{-}(\Gamma) \cup \{\alpha\}$ is consistent.*

Proof. Since we have at least one β such that $\Box\beta \in \Gamma$, $n^{-}(\Gamma) \neq \emptyset$. So suppose $n^{-}(\Gamma) \cup \{\alpha\}$ is not consistent. Then, for some $\beta_1, \ldots, \beta_n$ such that $\Box\beta_i \in \Gamma$, $\vdash \neg(\neg\beta_1 \wedge \ldots \wedge \neg\beta_n \wedge \alpha)$, that is, $\vdash \alpha \to (\beta_1 \vee \ldots \vee \beta_n)$. Using RqW we obtain $\vdash \Diamond\bot \to (\Box(\beta_1 \vee \ldots \vee \beta_n) \to \Box\alpha)$ and, by means of qZ, we get $\vdash \Diamond\bot \to ((\Box\beta_1 \wedge \ldots \wedge \Box\beta_n) \to \Box\alpha)$. From this we see that $\Box\alpha \in \Gamma$, making Γ inconsistent agains the hypothesis. ∎

Lemma 6.27. *Let Γ be a not C-distributive MCS such that $\Box\top \in \Gamma$ and, for some α, $\Box\alpha \in \Gamma$. Then $p(\Gamma) \cup \{\alpha\}$ is consistent*

Proof. As in **nocp**. ∎

6.2.5 Completeness for ncp

Not having opposite worlds simplifies our definition of a contrary MCS: an MCS Δ is said to be *contrary* if $\Diamond\bot \in \Delta$. Normal and perverse ones are as in Definition 6.10.

Lemma 6.28. *Let Γ be an MCS such that $\Diamond\bot \in \Gamma$ and $\Box\alpha \in \Gamma$. Then $p^{-}(\Gamma) \cup \{\neg\alpha\}$ is consistent.*

Proof. Since $\Diamond\bot \in \Gamma$, that is, $\neg\Box\top \in \Gamma$, $p^{-}(\Gamma) \neq \emptyset$. So suppose $p^{-}(\Gamma) \cup \{\alpha\}$ is not consistent. Then, for some $\beta_1, \ldots, \beta_n$ such that $\neg\Box\beta_i \in \Gamma$, $\vdash \neg(\beta_1 \wedge \ldots \wedge \beta_n \wedge \neg\alpha)$, that is, $\vdash (\beta_1 \wedge \ldots \wedge \beta_n) \to \alpha$. Using RqW we obtain $\vdash \Diamond\bot \to (\Box\alpha \to (\Box(\beta_1 \wedge \ldots \wedge \beta_n))$ and, by means of qY, we get $\vdash \Diamond\bot \to (\Box\alpha \to (\Box\beta_1 \vee \ldots \vee \Box\beta_n))$. From this it follows that $\vdash \Diamond\bot \to ((\neg\Box\beta_1 \wedge \ldots \wedge \neg\Box\beta_n) \to \neg\Box\alpha)$. But then $\neg\Box\alpha \in \Gamma$, making Γ inconsistent agains the hypothesis. ■

6.2.6 Completeness for np

Here we have only normal and perverse MCSs to deal with. The definitions have a slight alteration, and the first two lemmas below can be proven as in **nocp**.

Definition 6.29.

(a) Δ *is* normal *if either (i) for every α, $\Box\alpha \in \Delta$ or (ii) $\Box\top \in \Delta$, for at least some α, $\neg\Box\alpha \in \Delta$ and Δ is C-distributive.*

(b) Δ *is* perverse *if either (i) for every α, $\neg\Box\alpha \in \Delta$ or (ii) $\Box\top \in \Delta$ and Δ is not C-distributive.*

Lemma 6.30. *Let Γ be a C-distributive MCS containing $\Box\top$ and $\neg\Box\alpha$, for some formula α. Then $n(\Gamma) \cup \{\neg\alpha\}$ is consistent.*

Lemma 6.31. *Let Γ be a not C-distributive MCS such that $\Box\top \in \Gamma$ and $\Box\alpha \in \Gamma$. Then $p(\Gamma) \cup \{\alpha\}$ is consistent*

Lemma 6.32. *Every MCS is either normal or perverse.*

Proof. Let Δ be some MCS. If, for every α, $\Box\alpha \in \Delta$, then Δ satisfies the first requirement for normalcy. Obviously such a set will not contain any negated necessity, for it would be inconsistent. Similarly, if, for every α, $\neg\Box\alpha \in \Delta$, then Δ satisfies the first requirement for perversity. Else we have the situation where, say, $\Box\alpha \in \Delta$ and $\neg\Box\beta \in \Delta$. Now if Δ is C-distributive, it is normal. If not, since **np** has RM, from the tautology $\alpha \to \top$ we obtain $\Box\alpha \to \Box\top$, so $\Box\top \in \Delta$ and thus Δ is perverse. ■

7 Strict classical Sylvan logics

Let us go back to Sylvan's paper. His intention was to present relational semantics for several strict classical systems of modal logic. We say that a modal logic is *strict classical* if it includes the necessitations of all tautologies and is closed under the replacement of strict equivalents, that is, we have the following:

(□PL) $\{\Box\alpha : \alpha \text{ is a tautology}\}$,

(RRSE) $\alpha \leftrightarrow\!\!\!\!\!\prec \beta, \gamma \;/\; \gamma[\alpha/\beta]$.

Well-known strict modal logics are the Lewis systems **S1**–**S5**, as well as the systems **S0.9**° and **S0.9** of Lemmon's. In [4], Sylvan says that it is usually assumed that tautologies are necessary, that whatever is strictly implied by a necessity is also necessary, and that if something is necessary, it is also true. He thus presents a basic strict classical modal logic, called **S0.6**° in [5], which can be axiomatized adding the following rules to classical propositional logic (where $\alpha \prec \beta$ is defined as $\Box(\alpha \to \beta)$):

RP1. $\alpha \prec \beta, \Box\alpha \;/\; \Box\beta$,

RN$^+$. $\alpha \;/\; \Box\alpha$, α is a tautology,

RN$_*$. $\Box\alpha \;/\; \alpha$,

RS. $\alpha \prec \beta, \beta \prec \alpha \;/\; \Box\alpha \prec \Box\beta$.

Sylvan also presents an extension of this logic, **S0.7**°, obtained by adding the axiom K to **S0.6**°. Axiomatizations of **S0.9**° and **S1**° are also considered ([5], p.25).

Before going into the details of Sylvan's proposed semantics, a few words regarding strict equivalence, $\leftrightarrow\!\!\!\!\!\prec$. There are two ways to define it, which are not in general interchangeable (cf. [1], p.3):

(i) $\alpha \leftrightarrow\!\!\!\!\!\prec \beta =_{\text{df}} (\alpha \prec \beta) \wedge (\beta \prec \alpha)$,

(ii) $\alpha \leftrightarrow\!\!\!\!\!\prec \beta =_{\text{df}} \Box(\alpha \leftrightarrow \beta))$.

However, the two definitions are equivalent (cf. [1], p.5) in a logic that provides

(RF) $\Box(\alpha \wedge \beta) \;/\; \Box\alpha \wedge \Box\beta \qquad \Box\alpha \wedge \Box\beta \;/\; \Box(\alpha \wedge \beta)$.

Following Chellas and Segerberg's distinction, we can call a logic *strict$_T$ classical* ('T' for 'traditional way') if it is closed under the replacement of strict$_T$ equivalents (using (i) above), and *strict classical* if it is closed under the replacement of strict equivalents (using (ii) above). Accordingly, there are actually two versions of Sylvan's rule RS, namely:

RS$_T$. $\Box(\alpha \to \beta), \Box(\beta \to \alpha) \;/\; \Box(\Box\alpha \to \Box\beta)$.

RS. $\Box(\alpha \leftrightarrow \beta) \;/\; \Box(\Box\alpha \leftrightarrow \Box\beta)$.

This gives us two versions of the basic strict classical logic. We will rename Sylvan's version (which use RS$_T$) to **S0.6**$^\circ_T$, reserving the name **S0.6**° for the axiomatization with RS. (Later we will show that they are the same logic, but there are a few results we need to show first.) As for **S0.7**° and stronger logics, Chellas and Segerberg have shown ([1], p.5) that a logic that has $\Box$PL and K also has RF, and this is the case of **S0.7°**.

For $\mathbf{S0.6}^\circ_T$ and **S0.6°**, then, we will also have two versions of replacement of strict equivalents, that is:

(RRSE$_T$) $\Box(\alpha \to \beta)$, $\Box(\beta \to \alpha)$, $\gamma \,/\, \gamma[\alpha/\beta]$.

(RRSE) $\Box(\alpha \leftrightarrow \beta)$, $\gamma \,/\, \gamma[\alpha/\beta]$.

Proposition 7.1. *(i)* $\mathbf{S0.6}^\circ_T$ *is closed under* RRSE$_T$ *and (ii)* **S0.6°** *is closed under* RRSE*.*

Proof. By standard induction on the complexity of formulas. ■

Sylvan proposed semantics for $\mathbf{S0.6}^\circ_T$–**S1°** considering frames in which the normal points are distinguished, or normal and perverse. Let us use the following naming convention: **Nocp** is the logic of the class of all nocp-frames, if the set of normal points is distinguished, that is, validity and satisfiability are relativized to the normal points of a model. **NocP** is the logic of the same class of frames, but taking both normal and perverse points as distinguished. Analogously for other logics. Sylvan's proposal thus amounts to this:

S1°: **No**, the logic of the class of all no-frames, with normal points distinguished;

S0.9°: **Noc**, all noc-frames, with normal points distinguished;

S0.7°: **Nop**, all nop-frames, with normal points distinguished;

$\mathbf{S0.6}^\circ_T$: $\mathbf{NoP}^s$, all nop-frames, with normal *and* perverse points distinguished, and the extra requirement that R is serial with regard to perverse points (that is, for every $x \in P$ there is some y such that Rxy).[3]

As we know, however, his proposed semantics for **S0.9°**–**S1** failed — they characterize stronger logics. We can also show they are inadequate for $\mathbf{S0.6}^\circ_T$ and **S0.7°**. Which are, then, the logics of the corresponding classes of frames above? And how do we obtain semantics for Sylvan's strict logics?

In their paper, Chellas and Segerberg show us a way of obtaining strict classical logics out of classical ones — they are the *Lewis versions* of classical modal logics. If Λ is a logic, $\mathbf{Lew}(\Lambda)$ is the smallest set of formulas that includes Λ, $\Box\top$ and is closed under modus ponens and uniform substitution. In general, such logics *are not* classical, not being closed under RE.

An important theorem relating classical logics and their strict versions is the following one ([1], Lemma 9.1 on p.11):

[3]This extra requirement is need in order to make $\Box\top$ true also at perverse points — it fails if R is not serial.

Theorem 7.2 (Chellas–Segerberg). *Let Λ be a classical modal logic and α any formula.*

(a) $\vdash_\Lambda \Box\top \to \alpha$ *iff* $\vdash_{\mathbf{Lew(\Lambda)}} \alpha$*;*

(b) $\vdash_\Lambda \alpha$ *iff* $\vdash_{\mathbf{Lew(\Lambda)}} \Box\alpha$*, if Λ provides RnN$_*$;*

(c) $\mathbf{Lew(\Lambda)}$ *is strict classical and provides RN$_*$, if Λ provides RnN$_*$.*

In particular, if Λ provides RnN$_*$, we then have:

$$\vdash_\Lambda \alpha \to \beta \quad \text{iff} \quad \vdash_{\mathbf{Lew(\Lambda)}} \alpha \dashv \beta.$$

As we mentioned, they use mixed frames for the semantics, and validity for the Lewis versions of the prenormal logics is defined taking the set of normal points as distinguished: a formula is true in a model, say, if it is true at all normal points, and valid if true at all normal points of every model based on every frame.

Thus they show that **Lew(EK)** is **S0.9°**, and **Lew(EX)** is **S1°**. We can additionally show that the weaker logic, **Lew(EnK)** is Sylvan's **S0.7°**. Although in their paper a Lewis version of **E** is mentioned (Fig. 5 on p.21), a semantics for this logic is not provided. In fact, adding a set of normal worlds to a neighborhood frame and taking them as distinguished does not yield the Lewis version of **E**, but what we could call the *normal version* of **E**, **N(E)**. It is easy to see that K is a thesis of **N(E)** (because true at normal points), but not of **Lew(E)** (since **E** does not have nK). We will provide semantics for **Lew(E)** later below, but let us now return to the strict versions of Sylvan's logics.

As we know from Lemma 3.1, all our Sylvan logics are classical and provide RnN$_*$, so their Lewis versions are strict classical according to the theorem above.

If we add $\Box\top$ as an additional axiom, we need to avoid taking opposite and contrary worlds are distinguished, since it fails on them. $\Box\top$ also fails on perverse worlds per se — but if every perverse world sees at least another world, that is, if R is serial only regarding perverse worlds, then N also holds. This is what Sylvan does in his paper, when he distinguishes perverse worlds in a frame, suggesting that $\mathbf{S0.6}^\circ_T$ is $\mathbf{NoP}^s$.

Accordingly, we have two ways of validating $\Box\top$ in a frame: taking only the set of normal worlds as distinguished, or taking both normal and perverse worlds, restricting our attention to frames where R is P-serial.

In the next section, we discuss the strict logics obtained by distinguishing normal worlds. An overview of the strict logics discussed in this paper can be found in Figure 3.[4]

[4]The diagram also includes a few more logics, like $\mathbf{NocP}^s$, **N(E)**, **Lew(EnM)**, which, for reasons of space, we will not be able to examine here. Further work, in preparation, will present frames and models for these logics, and also discuss the results of placing restrictions like reflexivity, seriality etc. on the accessibility relation R.

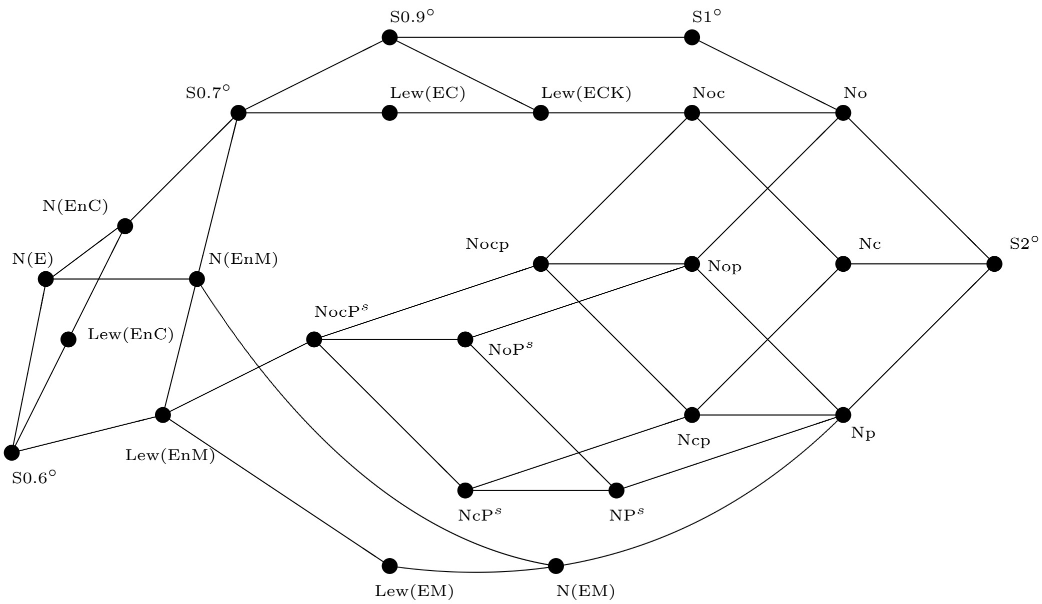

Figure 3: Strict modal logics

8 Distinguishing normal worlds

In this section we consider the logics **Nocp**, **Noc**, **Ncp**, **Nop**, **No**, **Nc**, **Np**, which are, respectively, the logics of nocp-frames, noc-frames, and so on, if we take the set of normal worlds as distinguished, that is, the validity and satisfiability of a formula in a frame are relativized to the normal worlds of the frame. If **n**x is a Sylvan logic, let us call the logic **N**x thus obtained its *normal version*.

Some of these logics are not Lewis versions of the logic on which they are based. Since normal worlds are distinguished, M, C, K, and X, are easily shown to be valid in **Nocp**, **Ncp**, **Nop** and **Np**. However, nK, nC and nX are not theses of **nocp**, **ncp**, **nop**, or **np** (recall, from Table 2, that they all fail at perverse worlds). But they should be, according to Theorem 7.2 above, if **Nocp** were a Lewis version of **nocp**, and so on. On the other hand, **Noc**, **No** and **Nc** are indeed the Lewis versions of **noc**, **no**, and **nc**, respectively, since they are all prenormal, having nK, and thus nM, nC and nX, as theses. Take **noc**, for example. Adding $\Box\top$ as an axiom does give us C, M, K and X. **Noc** is thus the Lewis version of **noc** — and, similarly, **Nc** and **No** are the Lewis versions of **no** and **nc**.

If **n**x is a Sylvan logic, we can axiomatize its normal version **N**x taking the closure under modus ponens and substitution of the following set of formulas:

(i) all theses of **n**x;

(ii) N, that is, $\Box\top$;

(iii) nC, that is, $\Box\top \to ((\Box\alpha \wedge \Box\beta) \to \Box(\alpha \wedge \beta))$.

Since nM is already provided in **nocp** (and all its extensions), adding N and nC will give us M, C, K and X.

Theorem 8.1. *Let* $\mathbf{n}x$ *be a Sylvan logic strongly determined by some class of frames. Then* $\mathbf{N}x$ *is strongly determined by the same class of frames if the set of normal points is distinguished.*

Proof. For correctness, all theorems of $\mathbf{n}x$ are valid at any normal point of a frame, and so are N and nC. And of course MP and US preserve validity.

For completeness, suppose Γ is an $\mathbf{N}x$-consistent set of formulas. Then $\Gamma \cup \{\text{N,nC}\}$ is $\mathbf{N}x$-consistent, too, and, since $\mathbf{n}x$ is contained in $\mathbf{N}x$, it is also $\mathbf{n}x$-consistent. So there is an $\mathbf{n}x$-MCS Δ containing Γ, N, and all instances of nC — and of C, since it contains N. By Lemma 6.18 and its analogous for other logics, all formulas of Δ are true at some point of some model in the appropriate class of frames. We must now show, for each logic, that Δ is normal for that logic.

Since Δ contains $\Box\top$, it is a normal MCS for **noc**, **no**, and **nc**.

Now the other four logics. Suppose first that, for every α, $\Box\alpha \in \Delta$. Then Δ is **nocp**-, **nop**-, **ncp**- and **np**-normal.

Suppose now this is not the case, that is, for at least one formula β, $\Box\beta \notin \Delta$, and so $\neg\Box\beta \in \Delta$. It is easy to see that Δ is C-distributive and thus a normal point in the canonical model for **nocp**-, **nop**-, **ncp**- and **np**, since contains all instantes of C: for no formulas $\alpha_1, \ldots, \alpha_n$ we have $\{\Box\alpha_1, \ldots, \Box\alpha_n\} \subseteq \Delta$ and $\Box(\alpha_1 \wedge \ldots \wedge \Box\alpha_n) \notin \Delta$. ■

Now two negative results regarding Sylvan's proposed semantics.

Proposition 8.2. $\mathbf{S1}^\circ$ *is* $\mathbf{Lew}(\mathbf{EX})$*, not* **No**.

Proof. That $\mathbf{S1}^\circ$ is not **No** was established in [2]. Chellas and Segerberg showed further that $\mathbf{S1}^\circ$ is $\mathbf{Lew}(\mathbf{EX})$. Now $\Box$C is valid in **No**, since C comes out true at every normal and opposite points. But $\Box$C is not a thesis of $\mathbf{S1}^\circ$, what can be shown by constructing an **EX**-model where C is falsifiable. On the other hand, **EX** is properly included in **no**; thus, **No** includes $\mathbf{S1}^\circ$. ■

Proposition 8.3. $\mathbf{S0.9}^\circ$ *is* $\mathbf{Lew}(\mathbf{EK})$*, not* **Noc**.

Proof. Chellas and Segerberg showed that $\mathbf{S0.9}^\circ$ is $\mathbf{Lew}(\mathbf{EK})$. Again, $\Box$C is **Noc**-valid, but not a thesis of $\mathbf{S0.9}^\circ$, since C is not **EK**-valid. On the other hand, **EK** is properly included in **noc**; thus, **Noc** includes $\mathbf{S0.9}^\circ$. ■

We will now discuss Sylvan's $\mathbf{S0.6}^\circ_T$, $\mathbf{S0.6}^\circ$, and $\mathbf{S0.7}^\circ$. But first a small preliminary result. In analogy to the distinction between the rules RS and RS_T, we can introduce two versions of the rule RE, namely:

(RE) $\alpha \leftrightarrow \beta \; / \; \Box\alpha \leftrightarrow \Box\beta$,

(RE_T) $\alpha \to \beta, \beta \to \alpha / \; \Box\alpha \to \Box\beta$

The proof of the following lemma is quite easy:

Lemma 8.4. *A logic is closed under RE iff it is closed under* RE_T.

Accordingly, the smallest classical logic **E** can be characterized either using RE, as it is usually done, or by means of RE_T. We will use this second version in the proof of the following lemma.

Lemma 8.5. *If* $\vdash_{\mathbf{E}} \alpha$ *then* $\vdash_{\mathbf{S0.6}^{\circ}_T} \Box\alpha$.

Proof. By induction on theorems. If α is a tautology, then $\vdash_{\mathbf{S0.6}^{\circ}_T} \Box\alpha$ by RN^+.

Suppose α was obtained by MP from β and $\beta \to \alpha$. By the induction hypothesis, $\vdash_{\mathbf{S0.6}^{\circ}_T} \Box\beta$, $\vdash_{\mathbf{S0.6}^{\circ}_T} \Box(\beta \to \alpha)$, that is, $\vdash_{\mathbf{S0.6}^{\circ}_T} \beta \dashv\!\!3\, \alpha$. By RP1, $\vdash_{\mathbf{S0.6}^{\circ}_T} \Box\alpha$.

Suppose α was obtained by US from β. By the induction hypothesis, $\vdash_{\mathbf{S0.6}^{\circ}_T} \Box\beta$ and, since $\mathbf{S0.6}^{\circ}_T$ provides US, $\vdash_{\mathbf{S0.6}^{\circ}_T} \Box\alpha$.

Suppose $\alpha = \Box\beta \to \Box\gamma$ and was obtained by RE_T from $\beta \to \gamma$ and $\gamma \to \beta$. By the induction hypothesis, $\vdash_{\mathbf{S0.6}^{\circ}_T} \Box(\beta \to \gamma)$ and $\vdash_{\mathbf{S0.6}^{\circ}_T} \Box(\gamma \to \beta)$, meaning that β and γ are $\mathrm{strictly}_T$ equivalent. Now $\Box\beta \to \Box\beta$ is a tautology; thus, by RN^+, $\vdash_{\mathbf{S0.6}^{\circ}_T} \Box(\Box\beta \to \Box\beta)$. And since $\mathbf{S0.6}^{\circ}_T$ is closed under the replacements of strict_T equivalents, $\vdash_{\mathbf{S0.6}^{\circ}_T} \Box(\Box\beta \to \Box\gamma)$. ∎

The same proposition, now for $\mathbf{S0.6}^{\circ}$.

Lemma 8.6. *If* $\vdash_{\mathbf{E}} \alpha$ *then* $\vdash_{\mathbf{S0.6}^{\circ}} \Box\alpha$.

Proof. The proof is analogous to the preceding lemma. We only need to consider RE instead of RE_T. So suppose $\alpha = \Box\beta \leftrightarrow \Box\gamma$ and was obtained by RE from $\beta \leftrightarrow \gamma$. By the induction hypothesis, $\vdash_{\mathbf{S0.6}^{\circ}} \Box(\beta \leftrightarrow \gamma)$, meaning that β and γ are strictly equivalent. Now $\Box\beta \leftrightarrow \Box\beta$ is a tautology; thus, by RN^+, $\vdash_{\mathbf{S0.6}^{\circ}} \Box(\Box\beta \leftrightarrow \Box\beta)$ and, by replacements of strict equivalents, $\vdash_{\mathbf{S0.6}^{\circ}} \Box(\Box\beta \leftrightarrow \Box\gamma)$. ∎

Lemma 8.7. *If* $\vdash_{\mathbf{E}} \alpha$ *then* $\vdash_{\mathbf{S0.6}^{\circ}_T} \alpha$ *and* $\vdash_{\mathbf{S0.6}^{\circ}} \alpha$.

Proof. Suppose $\vdash_{\mathbf{E}} \alpha$. By the two preceding lemmas, $\vdash_{\mathbf{S0.6}^{\circ}_T} \Box\alpha$ and $\vdash_{\mathbf{S0.6}^{\circ}} \Box\alpha$. Since both logics have RN_*, $\vdash_{\mathbf{S0.6}^{\circ}_T} \alpha$ and $\vdash_{\mathbf{S0.6}^{\circ}} \alpha$. ∎

Lemma 8.8. *If* $\vdash_{\mathbf{S0.6}^{\circ}_T} \alpha$ *then* $\vdash_{\mathbf{Lew(E)}} \alpha$.

Proof. If $\vdash_{\mathbf{S0.6}^{\circ}_T} \alpha$ then α is a tautology (in which case $\vdash_{\mathbf{Lew(E)}} \alpha$) or was obtained in a proof by some of the inference rules MP, US, RP1, RN^+, RN_* or RS_T. Since $\mathbf{Lew}(\mathbf{E})$ is closed under MP and US, we need to show that it provides the remaining rules.

For RP1, suppose $\vdash_{\mathbf{Lew(E)}} \Box(\alpha \to \beta)$, and $\vdash_{\mathbf{Lew(E)}} \Box\alpha$. By Theorem 7.2, $\vdash_{\mathbf{E}} \alpha \to \beta$, and $\vdash_{\mathbf{E}} \alpha$. Then $\vdash_{\mathbf{E}} \beta$ by MP and, again by Theorem 7.2, $\vdash_{\mathbf{Lew(E)}} \Box\beta$.

For RN$^+$, suppose α is a tautology. Then it is a theorem of both **E** and **Lew(E)**. By Theorem 7.2, $\vdash_{\mathbf{Lew(E)}} \Box\alpha$.

For RN$_*$, suppose $\vdash_{\mathbf{Lew(E)}} \Box\alpha$. Then $\vdash_{\mathbf{E}} \alpha$ by Theorem 7.2, and thus $\vdash_{\mathbf{E}} \Box\top \to \alpha$. Using again Theorem 7.2, we get $\vdash_{\mathbf{Lew(E)}} \alpha$.

For RS$_T$, suppose $\vdash_{\mathbf{Lew(E)}} \Box(\alpha \to \beta)$ and $\vdash_{\mathbf{Lew(E)}} \Box(\beta \to \alpha)$. Then $\alpha \to \beta$, $\beta \to \alpha$ are theorems of **E**, and thus, by RE$_T$, so is $\Box\alpha \to \Box\beta$. But then $\Box(\Box\alpha \to \Box\beta)$ is a theorem of **Lew(E)** by Theorem 7.2. ■

Lemma 8.9. *If* $\vdash_{\mathbf{S0.6^\circ}} \alpha$ *then* $\vdash_{\mathbf{Lew(E)}} \alpha$.

Proof. Analogous to the preceding lemma. We only need to consider RS instead of RS$_T$. So suppose $\vdash_{\mathbf{Lew(E)}} \Box(\alpha \leftrightarrow \beta)$. Then $\alpha \leftrightarrow \beta$ is a theorem of **E**, and thus, by RE, so is $\Box\alpha \leftrightarrow \Box\beta$. But then $\Box(\Box\alpha \leftrightarrow \Box\beta)$ is a theorem of **Lew(E)** by Theorem 7.2. ■

Theorem 8.10. $\vdash_{\mathbf{Lew(E)}} \alpha$ *iff* $\vdash_{\mathbf{S0.6^\circ_T}} \alpha$ *iff* $\vdash_{\mathbf{S0.6^\circ}} \alpha$.

Proof. Suppose $\vdash_{\mathbf{Lew(E)}} \alpha$. We have three cases. (i) If α is a theorem of **E**, by Lemma 8.7 $\vdash_{\mathbf{S0.6^\circ_T}} \alpha$ and $\vdash_{\mathbf{S0.6^\circ}} \alpha$. (ii) If $\alpha = \Box\top$, then $\vdash_{\mathbf{S0.6^\circ_T}} \alpha$ and $\vdash_{\mathbf{S0.6^\circ}} \alpha$, since both logics are closed under RN$^+$. (iii) α was obtained by MP or US, the result also holds, since both logics have these rules.

The other direction follows from Lemmas 8.8 and 8.9. ■

From the above theorem we gather that **Lew**(**E**), $\mathbf{S0.6^\circ_T}$ and $\mathbf{S0.6^\circ}$ are actually the same logic. We get thus the following result, against Sylvan's contention:

Proposition 8.11. $\mathbf{S0.6^\circ}$ *is* **Lew**(**E**)*, not* $\mathbf{NoP}^s$.

Proof. Consider $\Box$nM and suppose it is not a thesis of $\mathbf{NoP}^s$. Then $\Box$nM fails at a normal or perverse point x. If x is normal, there is an accessible world where nM fails. But nM holds in all points, so that can't be. If x is perverse, nM fails at all accessible points. Since we are supposing that R is serial regarding to elements of P, there is at least one accessible world y, in which nM fails. And again that can't be. So $\Box$nM is $\mathbf{NoP}^s$-valid. Now, if it were a theorem of $\mathbf{S0.6^\circ}$, that is, **Lew**(**E**), then nM would be a theorem of **E** and thus valid. But nM is not **E**-valid. ■

Finally, let us consider $\mathbf{S0.7^\circ}$. This logic is obtained by adding K to $\mathbf{S0.6^\circ}$. Now **EnK** has nK, so **Lew(EnK)**, its Lewis version, has K. We can show the following:

Theorem 8.12. $\vdash_{\mathbf{Lew(EnK)}} \alpha$ *iff* $\vdash_{\mathbf{S0.7^\circ}} \alpha$.

Proof. The overall strategy is the same as in the proof of Theorem 8.10. ■

Proposition 8.13. $\mathbf{S0.7^\circ}$ *is* **Lew**(**EnK**)*, not* **Nop**.

Proof. That **S0.7**° and **Lew(EnK)** are the same logic follows from the preceding theorem. Now consider nK: it is not valid in **nop**, so $\Box$nK is not valid in **Nop**. On the other hand, nK is valid in **EnK**, making $\Box$nK valid in **Lew(EnK)**. On the other hand, qW is a theorem of **nocp** and thus of **nop**, making $\Box$qW is not a thesis of **Nop**. However, as we saw on p.10 above, qW is not valid in **EnK**, so $\Box$qW is not a thesis of **Lew(EnK)**. ■

9 Final remarks

In this paper, inspired by a paper of Richard Sylvan's, we presented a few so-called *Sylvan logics*, that is, the logics of relational frames containing one or more kinds of non-normal worlds. We proved determination theorems for several of the logics thus obtained, and discussed theire relation to classical systems of modal logic. In a second part of the paper, drawing on work by B. Chellas and K. Segerberg, we considered strict logics obtained by taking, for instance, the set of normal worlds as distinguished. We showed, among other results, that Sylvan's claims about the semantics for **S0.6**° and **S0.7**°, besides those for **S0.9**° and **S1**°, are also inadequate, determining, in fact, stronger logics.

However, many questions remain, which could not be adressed here. For instance, we did not discuss Lewis versions of **nocp**, **ncp**, **nop**, and **np**, neither the logics of the class of frames serial with regard to P-points, which we could call $\mathbf{nocp}^s$, $\mathbf{ncp}^s$, $\mathbf{nop}^s$, and $\mathbf{np}^s$, and their corresponding strict logics. Besides this, adding other conditions on the relation R will give rise to some interesting results, at least from the technical point of view. A related question is whether such weak logics can find interesting applications, philosophical or other: a question for further work. As Sylvan said ([5], p.22), regarding weak modal logics, "philosophical virtue lies in weakness".[5]

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[5] Parts of this paper were presented at the *VII Principia International Symposium* and *XV Encontro Nacional de Filosofia da ANPOF*; thanks to the audience for the comments. Many thanks also to this Journal's two anonymous referees for very helpful cricitism and suggestions.

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