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# Quasi-Truth and Possible Worlds

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**Abstract**

The Quasi-Truth theory offers an important formal approach to understand the scientific knowledge partiality – through the so called "partial structures" – so that it is a theory capable of dealing with the notion of "pragmatic truth". Some important notions that we find in scientific works, that escape from the formal treatment offered by the theory of quasi-truth, are modal ones such as, for example, *necessity*, *possibility* and *counterfactual statements*. In this paper we intend to develop a possible worlds semantics for the Quasi-Truth theory, by preserving the approach of both partiality of scientific and pragmatic truth, but also offering an interpretation for modal operators. Within this formalism, we shall show that we can get several modal systems. Generally speaking, our aim is to reach to a theory that comprises the concept of quasi-truth and that is able to treat modality as is applied to scientific theories.

**Keywords:** Quasi-Truth; Partial Structures; Possible Worlds; Modal Semantic.

## Introduction

The Quasi-Truth theory offers a formal approach capable of dealing with two important aspects of a scientific theory: (1) the partiality of scientific knowledge, *i.e.*, contexts where complete information are not available; and (2) the pragmatic notion of truth (also known as *quasi-truth*), *i.e.*, given the difficulty of determining truth in partial contexts, the use of pragmatic (or partial) truth is appropriate (*e.g.*, for pragmatic probability, acceptance of theories in science, nature of scientific reasoning, and the role of "models" in science).[1] However, there are two other important aspects of scientific theory that Quasi-Truth theory does not address to: (3) the application of modality in the scientific context, for example, through the use of terms like "necessary", "possible" and "contingent"; and (4) use of counterfactual sentences, for example in statements

[1] *Cf.* [15]; [2]; [5]; [12]; [4]; [3].

such as: if the earth had a mass $n$ times smaller, the moon would escape its orbit.[2]

The general aim of this paper is to deal with the modality in the quasi-truth theory, offering a theory based on the formal structures of the quasi-truth (preserving the interpretation of the scientific knowledge partiality and the notion of quasi-truth) and, on such formalism, to develop a semantics capable of interpreting modal operators and counterfactual sentences. At first, I will explain what is *quasi-truth*, introducing its necessary formalism and the pertinent modifications to this work. After, I will offer a possible-worlds semantics on the formal structure of quasi-truth with the needed adaptations. Finally, I will briefly discuss the relevant consequences (philosophical and formal) that we can derive from the resulting theory and address some topics for further discussion.

# 1 Quasi-Truth

In order to investigate some area of knowledge $\Delta$ (*e.g.*, the movement of celestial bodies), scientists generally formulate conceptual frameworks that allow one to systematize and organize the information obtained about $\Delta$ – given them, even if implicitly, some appropriate postulates. Usually we call this conceptual frameworks "theories", and with them we can represent the domain of entities treated by $\Delta$ through the set $D$. That is, we associate $\Delta$ with a set $D$ of entities (containing both *real* and *ideals* objects), in a way that we investigate and study the relations among the elements of $D$ through the theory. The main aspect here will be the *incompleteness* of information. It is common to face situations in which, given a certain n-ary relation $R_n$ (defined over $D^n$), we do not know if all the elements of $D$ (or all n-tuples) are or not $R_n$-related. How can we formally interpret this partiality of scientific knowledge? For this, Mikenberg, da Costa and Chuaqui [15] developed the theory of Quasi-Truth – or, as it is also known, the theory of "Pragmatic Truth".

Let $\mathfrak{T}$ be a properly formalized scientific theory in a first-order language $\mathfrak{L}$. Through this formulation we will see that $\mathfrak{T}$ has as axioms: (LA) the axioms of the underlying logic (axioms of $\mathfrak{L}$); and (EA) the specific axioms of the scientific theory that we aim to treat.[3] Now we can define the notion of *Partial Structures* that we aim to capture the idea of partiality of scientific knowledge.[4]

[2] *Cf.* [7]; [8]; [11]; [14].

[3] The axiomatic approach to scientific theories faces some difficulties. However, we will not address the meta-theoretical problems of how we should (formally) analyze scientific theories. For such discussion, *see* [5, p.21-8] and [13].

[4] Note that the following metalanguage is built on *Zermelo-Fraenkel* set theory, with his usual symbolism.

**Definition 1.1 (Partial Structures)** *Let $\mathcal{A} = \langle D, R_i \rangle_{i \in I}$ be a structure that:*

*(a)* $D \neq \emptyset$

*(b)* $(R_i)_{i \in I}$ *it is a family of partial relations over D, where* $R_n = \langle R_n^1, R_n^2, R_n^3 \rangle$*, and that:*

*(b.1)* $R_n^1$ *is the set of n-tuples that we know that belongs to* $R_n$

*(b.2)* $R_n^2$ *is the set of n-tuples that we know that do not belongs to* $R_n$

*(b.3)* $R_n^3$ *is the set of n-tuples for which it is not defined whether they belongs or not to* $R_n$

*(b.4)* $R_n^1 \bigcup R_n^2 \bigcup R_n^3 = D^n$

*We call $\mathcal{A}$ as "Partial Structure".*[5]

Note that the partiality of a structure has an epistemic character, representing an incompleteness of our knowledge about the domain $\Delta$ that we are investigating.[6]

Let $\alpha$ be a sentence of $\mathfrak{T}$. How can we assign truth values to $\alpha$ in order to preserve the partiality of the scientific context? An important aspect of Tarski's characterization of truth is that a sentence of a language is true or false only in relation to a given *interpretation* in some *structure*.[7] Similarly, a sentence can be quasi-true (or quasi-false) only in relation to a specific kind of structure. The notion of quasi-truth uses the Tarski's characterization of truth. However, Tarski's characterization employs only *total* structures, where the relations are *usual* (not-partial) – that is, they are defined to every elements in the domain. If the structures are total, we lose what we are looking for: the partiality. To make a link between partial structures and total structures (thus being able to characterize truth in *Tarskian-Style*) we need the auxiliary notion of *Simple Pragmatic Structure*, which is a partial structure to which we incorporate a third component, namely, a set $P$ of sentences of $\mathfrak{T}$ that we assume as true.[8] The sentences of $P$ are those that can express empirically decidable statements or general sentences expressing *laws* described by the theory that we are dealing with [5, p.18-19].

**Definition 1.2 (Simple Pragmatic Structures)** *Let $\mathcal{A}^s = \langle D, R_i, P \rangle_{i \in I}$ be a structure to $\mathfrak{T}$ that:*

[5]If $R_n^3 = \emptyset$, then $R_n$ is a usual *n-ary* relation, identified by $R_n^1$.

[6]If the partiality of the relations has an epistemic character, the Quasi-Truth theory will face some interesting philosophical problems; for example, how can the Quasi-Truth theory represent the ontological aspects of a scientific theory? For the sake of brevity we will not deal with this problems here.

[7]*Cf.* [9, p. 143-156].

[8]*Cf.* [10, p.88].

*(a)* $D \neq \emptyset$

*(b)* $(R_i)_{i \in I}$ *is a family of partial relations over D.*

*(c)* $P$ *is a set of accepted sentences of* $\mathfrak{T}$.

*We call* $\mathcal{A}$ *as a "Simple Pragmatic Structure" to* $\mathfrak{T}$ *.*

In order to introduce the modal notions of *necessity* and *possibility*, according to our purposes, we need to make a slight modification in the definition of Simple Pragmatic Structure. Accordingly to the previously definition, a Simple Pragmatic Structure incorporates a set $P$ of sentences of $\mathfrak{T}$ which express empirically decidable statements, as also general sentences expressing *laws* described by the theory that we are dealing with. The main modification will be in the way we will introduce the sentences that we assume as true.

**Definition 1.3 (Modal Pragmatic Structures)** *Let* $\mathcal{A}^m = \langle D, R_i, G, Q \rangle_{i \in I}$ *be a structure to* $\mathfrak{T}$ *that:*

*(a)* $D \neq \emptyset$

*(b)* $(R_i)_{i \in I}$ *is a family of partial relations over D.*

*(c)* $G$ *is a set of axioms or theorems of* $\mathfrak{T}$.

*(d)* $Q$ *is a set of empirically decided sentences of* $\mathfrak{T}$*, taken to be true.*[9]

*We call* $\mathcal{A}^m$ *as a "Modal Pragmatic Structure" to* $\mathfrak{T}$.

It should be noted that, while in a Simple Pragmatic Structure the set $P$ of sentences encompasses both theorems as well as empirically decidable sentences, in a Modal Pragmatic Structure we distinguish those sentences taken as theorems (belonging to set $G$) from those that are empirically decidable (belonging to set $Q$). When we observe the sets $G$ and $Q$ that belongs $\mathcal{A}^m$ and the set $P$ that belongs to $\mathcal{A}^s$, we can notice that: $G \bigcup Q = P$.

As stated earlier, in order to define the quasi-truth we will need to extend the partial structure that we have for a *total* structure (and in it we will define the notion of truth in *tarskian way*). The connecting link between these two structures would be the *Simple Pragmatic Structures*, but for our purposes we will use the *Modal Pragmatic Structures.* Let us see what is a *total structure* that extends a modal pragmatic structure.

[9]A further problem we can face is how can we define what is an "empirically decidable" sentence. I will assume that these sentences are such that scientists only determine its truth-value through experiments.

**Definition 1.4 (Total Structures)** *Let $B = \langle D', R_i', \rho\rangle_{i\in I}$ be a structure to $\mathfrak{T}$ that:*

*(a)* $D' \neq \emptyset$

*(b) The relations $(R_i')_{i\in I}$ defined over $D'$ extend their correspondent partial relations $(R_i)_{i\in I}$ of modal pragmatic structure $\mathcal{A}^m$, defining them for all n-tuples of elements of $D'$.*

*(c) $\rho$ an interpretation function that:*

*(c.1) For all constant c of $\mathfrak{T}$, $\rho(c)$ belongs to D'.*

*(c.2) For all n-ary relational symbol R of $\mathfrak{T}$, $\rho(R)$ will be a n-ary relation over D'.*

*(c.3) For all n-ary function f of $\mathfrak{T}$, $\rho(f)$ will be a n-ary function over D'.*

*We call $\mathcal{B}$ a "Total Structure" to $\mathfrak{T}$, that all of n-ary relation is defined to all n-tuples of elements belonging to $D'$.*

Thus we have the notion of *Modal Pragmatic Structures* and *Total Structures.* We can now explain how we extend a *Partial Structure* to a *Total Structure.*

**Definition 1.5 ($\mathcal{A}^m$-Normal Structures)** *Let $\mathfrak{T}$ be an axiomatized theory of a first-order language, $\mathcal{A}^m$ a modal pragmatic structure and B a total structure, where $\mathfrak{T}$ is interpreted. $\mathcal{B}$ will be an $\mathcal{A}^m$-normal structure if the follow properties are satisfied:*

*(a)* $D = D'$

*(b) The total relations $(R_i')_{i\in I}$ of $\mathcal{B}$ extend the corresponding partial relations $(R_i)_{i\in I}$ of $\mathcal{A}^m$.*

*(c) If c is an individual constant of $\mathfrak{T}$, then c is interpreted in $\mathcal{A}^m$ and $\mathcal{B}$ by the same element.*

*(d) If $\alpha \in G$, then $\mathcal{B}$ satisfies $\alpha$. That is, every sentence that belongs to G is valid in the structure $\mathcal{B}$, what we denote by:*

$$B \models \alpha.$$

Note that through the previous definition only the sentences that belong to $G$ are satisfied in every $\mathcal{A}^m$-Normal structures. The same will not happen to the sentences that belong to $Q$, which may or may not be satisfied by $\mathcal{A}^m$-Normal structures.[10] In the same way, for each partial structure $\mathcal{A}^m$ we can obtain several $\mathcal{A}^m$-Normal Structures. Once we introduce the set of sentences $G$ (which will be axioms or theorems of $\mathfrak{T}$), we restrict which are the Total Structures that extend $\mathcal{A}^m$ – that is, which Total Structures are $\mathcal{A}^m$-Normal Structures. We have the necessary tools to formulate the standard definition of quasi-truth, which would follow as:

**Definition 1.6 (*Standard* Quasi-Truth)** *Let $\mathfrak{T}$ be an axiomatized theory in a first-order language, $\mathcal{A}^s$ a simple pragmatic structure, where $\mathfrak{T}$ is interpreted, and $\mathcal{B}$ an $\mathcal{A}^s$-normal structure. A sentence $\alpha$ is quasi-true in a structure $\mathcal{A}^s$, according to a structure $\mathcal{B}$, if and only if, $\alpha$ is true in $\mathcal{B}$ in the usual sense. Otherwise, we say that $\alpha$ in quasi-false.*

We should note that this is the standard definition of quasi-truth, using the notion of *Simple Pragmatic Structures*. In the definition of $\mathcal{A}^s$-Normal Structures, condition (d) is such that $\alpha \in P$, that is, we will not work with the set $G \in \mathcal{A}^m$, but only with the set $P \in \mathcal{A}^s$. We can appropriately adapt it to Modal Pragmatic Structures, however, we will formulate this notion in another way, using the accessibility relation between structures.

## 2 Modality and Possible Worlds Semantics

Once we made the necessary modifications in the Quasi-Truth theory by defining the notion of *Modal Pragmatic Structures*, we can now extend the language of $\mathfrak{T}$ with modal operators. Let $\mathfrak{T}_m$ be an extension of $\mathfrak{T}$ with the symbol "$\Box$", that we assume as primitive and which the intuitive interpretation will be as *necessity operator*. We will define the symbol "$\Diamond$", that we will intuitively interpret as *possibility operator*, from the necessity operator as follows:

$$\Diamond\alpha =_{\text{def}} \neg\Box\neg\alpha$$

Now we can offer a Kripkean-style semantics.[11] Intuitively, I want to make the $\mathcal{A}^m$-normal structures – which extend the modal pragmatic structure $\mathcal{A}^m$ – being such as "possible-worlds" in a modal semantic. The very structure $\mathcal{A}^m$, however, will not be itself a *world* (since it is a partial structure). It will be, therefore, the initial *node* of our frame.

[10] The usual characterizations of the quasi-truth [2] uses the *Simple Pragmatic Structures*, so that the $\mathcal{A}^s$-Normal must therefore satisfy all sentences $\alpha \in P$. Thus, both the sentences belonging to $G$ and $Q$, in a given *Modal Pragmatic Structure*, would be satisfied.

[11] The method we will use follows a formal semantics for quantifiers with constant domain frames – also with appropriate adjustments. *Cf.* [7, Cap.4].

**Definition 2.1 (Frame)** *Let* $\mathcal{K} = \langle \mathcal{A}^m, \mathfrak{W}, \mathfrak{D}, \mathfrak{R}, \ell \rangle$ *be a structure that:*

*(a)* $\mathcal{A}^m$ *is a modal pragmatic structure.*

*(b)* $\mathfrak{W}$ *is a set of* $\mathcal{A}^m$*-normal structures — designated by* $w, w_1, ..., w_n$ *the elements of* $\mathfrak{W}$ *and which we can call "worlds".*

*(c)* $\mathfrak{D}$ *is the domain of* $\mathcal{K}$ *that* $\mathfrak{D} = D \in \mathcal{A}^m$. *That is, the domain of* $\mathcal{K}$ *is constant, and equal to domain of the partial structure* $\mathcal{A}^m$*, consequently, equal to all of the* $\mathcal{A}^m$*-normal structures.* [12]

*(d)* $\mathfrak{R}$ *is an accessibility relation between structures, defined as* $\mathfrak{R} = \mathfrak{R}' \cup \mathfrak{R}''$*, that:*

*(d.1)* $\mathfrak{R}' = \{\langle \mathcal{A}^m, w \rangle : w \in \mathfrak{W}\}$ *and*

*(d.2)* $\mathfrak{R}'' = \{\langle w, w_1 \rangle : w, w_1 \in \mathfrak{W}\}$.

*That is,* $\mathfrak{R}$ *is a binary relation that relates both* $\mathcal{A}^m$ *with elements of* $\mathfrak{W}$ *(through* $\mathfrak{R}'$*), and elements of* $\mathfrak{W}$ *itself (through* $\mathfrak{R}''$*) as well.*

*(e)* $\ell$ *is an interpretation function over* $\mathcal{K}$*, where* $\ell$ *assign for each n-ary relational symbol* $R$ *and for each* $\mathcal{A}^m$*-normal structure* $w$ *(that* $w \in \mathfrak{W}$*) some n-ary relation over the domain* $\mathfrak{D}$.

*We call* $\mathcal{K}$ *a "Frame" to* $\mathfrak{T}_m$.

**Definition 2.2 (Valuation)** *Let* $\mathcal{K}$ *be a frame to* $\mathfrak{T}_m$. *A valuation in a frame* $\mathcal{K}$ *it is an assignment* $v$ *which designates, for each free variables* $x$*, some element* $v(x)$ *that belongs to* $\mathfrak{D}$.

We can then offer the notion of satisfiability, being denoted as:

$$\mathcal{K}, w \models_v \alpha$$

where $\mathcal{K}$ is a first-order frame with constant domain, $w$ an $\mathcal{A}$-normal structure that belongs to $\mathfrak{W}$, $\alpha$ a sentence, possibly with free variables, and $v$ a valuation. We say that a sentence $\alpha$ is true in $w$ of the structure $\mathcal{K}$ in relation to a valuation $v$, where $v$ tell us which value will be assign to every free variables.

[12] Note that we could define $\mathfrak{D}$ as a *domain function*, such that $\mathfrak{D}$ associates each world $w \in \mathfrak{W}$ with a non-empty set $\mathfrak{D}(w)$, called the "domain of $w$". According to the definition of $\mathcal{A}^m$-Normal Structures, every $\mathcal{A}^m$-Normal Structure has the same domain as the Modal Pragmatic Structure $\mathcal{A}^m$. Therefore, we must impose such a condition on our frame $\mathcal{K}$, making it a constant domain. We impose this condition by establishing that $\mathfrak{D}(w_1) = \mathfrak{D}(w_2)$ to every $w_1, w_2 \in \mathfrak{W}$. The characterization of $\mathfrak{D}$ as a *domain function* (and not just a set) will be enlightening later on, when dealing with the Barcan's formula. However, if we follow this approach from now on, the subsequent definitions (for example, valuation and truth in a *frame*) should be changed, bringing unnecessary complications. *Cf.* [6]; [7, Cap.4].

**Definition 2.3 (Variant)** *Let $v$ and $u$ be two valuations. We say that valuation $u$ is $x$-variant of $v$ if $v$ and $u$ agree in all variables except the possible variable $x$.*

From these definitions we can offer the notion of truth in a frame for $\mathfrak{T}_m$. Note that *truth*, in this sense, is the Tarski's notion of truth extended to a possible worlds semantics.

**Definition 2.4 (Truth in a frame)** *Let $\mathcal{K} = \langle \mathcal{A}^m, \mathfrak{W}, \mathfrak{D}, \mathrm{R}, \ell \rangle$ be a first-order frame with constant domain. For each $w \in \mathfrak{W}$ and each valuation $v$ in $\mathcal{K}$:*

*(1) If $R$ is a n-ary relational symbol, then $\mathcal{K}$, $w \models_v R(x_1, ..., x_n)$ iff $\langle v(x_1), ..., v(x_n) \rangle \in \ell(R, w)$.*

*(2) $\mathcal{K}, w \models_v \alpha$ iff $\mathcal{K}, w \not\models_v \neg\alpha$*

*(3) $\mathcal{K}, w \models_v \alpha \rightarrow \beta$ iff $\mathcal{K}, w \not\models_v \alpha$ or $\mathcal{K}, w \models_v \beta$*

*(4) $\mathcal{K}, \mathcal{A}^m \mathfrak{R} w \models_v \alpha$ iff $\mathcal{A}^m \mathfrak{R} w$ and $\mathcal{K}, w \models_v \alpha$*

*(5.a) $\mathcal{K}, w \models_v \Box\alpha$ iff for all $w_1 \in \mathfrak{W}$, if $w \mathfrak{R} w_1$ then $\mathcal{K}, w_1 \models_v \alpha$.*

*(5.b) $\mathcal{K}, \mathcal{A}^m \mathfrak{R} w \models_v \Box\alpha$ iff for all $w_1 \in \mathfrak{W}$, if $\mathcal{A}^m \mathfrak{R} w_1$ then $\mathcal{K}, w_1 \models_v \alpha$.*

*(6.a) $\mathcal{K}, w \models_v \Diamond\alpha$ iff for some $w_1 \in \mathfrak{W}$, $w \mathfrak{R} w_1$ and $\mathcal{K}, w_1 \models_v \alpha$.*

*(6.b) $\mathcal{K}, \mathcal{A}^m \mathfrak{R} w \models_v \Diamond\alpha$ iff for some $w_1 \in \mathfrak{W}$, $\mathcal{A}^m \mathfrak{R} w_1$ and $\mathcal{K}, w_1 \models_v \alpha$.*

*(7) $\mathcal{K}, w \models_v (\forall x)\alpha$ iff for all $x$-variant $u$ of $v$ in $\mathcal{K}$, $\mathcal{K}, w \models_u \alpha$*

*(8) $\mathcal{K}, w \models_v (\exists x)\alpha$ iff for some $x$-variant $u$ of $v$ in $\mathcal{K}$, $\mathcal{K}, w \models_u \alpha$*

We should make some important notes about the previous definition. In condition (1), $R$ is a total relation in $w$ (which is a total $\mathcal{A}^m$-normal structure) in which R extends a partial relation of $\mathcal{A}^m$. In condition (2) we have the notion of satisfiability in $\mathcal{A}^m$-Normal structures for $\mathfrak{T}_m$. In $\mathcal{A}^m$, which is a partial structure, we will have only the notion of quasi-truth, which we will shall define through the relation $\mathfrak{R}$ of accessibility. In (5.a) the modal operator $\Box$ is defined to an accessibility relation between $\mathcal{A}^m$-Normal structures. Recalling that the accessibility relation $\mathfrak{R}$ was defined as $\mathfrak{R} = \mathfrak{R}' \cup \mathfrak{R}''$, where $\mathfrak{R}'$ relates the $\mathcal{A}^m$ structures with $\mathcal{A}^m$-Normal structures, and $\mathfrak{R}''$ a relation between $\mathcal{A}^m$-Normal structures. Therefore, (5.a) determines the $\mathfrak{R}''$ accessibility relation. On the other hand, in (5.b) the modal operator $\Box$ is defined for the $\mathfrak{R}'$ accessibility relation, between $\mathcal{A}^m$ and $\mathcal{A}^m$-Normal structures. The same happens for the modal operator $\Diamond$ in (6.a) and (6.b). We can now offer a definition of truth (in a tarskian-style) in a $\mathcal{A}^m$-Normal structure.

**Definition 2.5 (Truth in the $\mathcal{A}^m$-normal structures)** *Let $\mathcal{K} = \langle \mathcal{A}^m, \mathfrak{W}, \mathfrak{D}, \mathfrak{R}, \ell \rangle$ be a frame with constant domain and $w \in \mathfrak{W}$. For each sentence $\alpha$, if $\mathcal{K}, w \models_v \alpha$ for some valuation $v$ in $\mathcal{K}$, then $\mathcal{K}, w \models_v \alpha$ for every valuation $v$ in $\mathcal{K}$ and reciprocally. We abbreviate these cases to $\mathcal{K}, w \models \alpha$ and say that $\alpha$ is true in $w$.*

After defining the notion of truth in the $\mathcal{A}^m$-normal structures, we can define the notion of quasi-truth, for $\mathfrak{T}_m$, through the $\mathfrak{R}$ accessibility relation between structures.

**Definition 2.6 (Quasi-Truth)** *Let $\mathcal{K}$ be a frame with constant domain to $\mathfrak{T}_m$ and $\Gamma$ the set of sentences of modal pragmatic structures $\mathcal{A}$. We can define a metalinguistic function that maps the ordered-pairs of $\Gamma$ and $\mathfrak{W}$ and assign a value 0 or 1:*

$$QV : \langle \Gamma, \mathfrak{W} \rangle \to \{0, 1\}$$

*We use that function to define the notion of quasi-truth – let $\alpha \in \Gamma$ and $w \in \mathfrak{W}$:*

$$QV(\alpha)_w = 1 \Leftrightarrow \mathcal{K}, \mathcal{A}\mathfrak{R}w \models \alpha$$

$$QV(\alpha)_w = 0 \Leftrightarrow \mathcal{K}, \mathcal{A}\mathfrak{R}w \not\models \alpha$$

*That is, $\alpha$ is quasi-true in $\mathcal{A}^m$, accordingly to an $\mathcal{A}^m$-Normal structure, if and only if, $\mathcal{A}^m$ is $\mathfrak{R}$-related with $w$ and $\mathcal{K}, w \models \alpha$, otherwise $\alpha$ is quasi-false.*

Note that through the previous definition, there may be a Modal Pragmatic Structure $\mathcal{A}^m$, such that given a frame with constant domain $\mathcal{K}$ and two $\mathcal{A}^m$-Normal structures, $w_1$ and $w_2$, a sentence $\alpha$ may be quasi-true in $\mathcal{A}^m$ according to $w_1$, and quasi-false in $\mathcal{A}^m$ according to $w_2$. That is, $QV(\alpha)_{w_1} = 1$ and $QV(\alpha)_{w_2} = 0$.

**Theorem 2.7** *Let $\mathcal{K} = \langle \mathcal{A}^m, \mathfrak{W}, \mathfrak{D}, \mathfrak{R}, \ell \rangle$ be a frame with constant domain to $\mathfrak{T}_m$ and $\mathcal{A}^m = \langle D, R_i, G, Q \rangle$ a modal pragmatic structure , that $G$ is the set of sentences that are axioms or theorems of $\mathfrak{T}$. For every sentence $\alpha \in G$ and all $\mathcal{A}^m$-normal structure $w \in \mathfrak{W}$, it follows that:*

$$\mathcal{K}, w \models \alpha$$

**Proof.** Let $\mathcal{A}^m$ be a Modal Pragmatic Structure and $\alpha$ a sentence, such that $\alpha \in G$. Given the definition of $\mathcal{A}^m$-Normal structures (Def. 1.5, p. 5), every $\mathcal{A}^m$-Normal structure $\mathcal{B}$, $\mathcal{B} \models \alpha$. Let $\mathcal{K}$ be a frame with constant domain and $\mathfrak{W}$ the set of every $\mathcal{A}^m$-Normal structures, for every $w \in \mathfrak{W}$: $\mathcal{K}, w \models \alpha$. ∎

**Corollary 2.8 (Necessitation Rule)** *Let $\mathcal{K}$ be a frame with constant domain, every theorem of $\mathfrak{T}_m$ (i.e., for all sentences $\alpha \in G$) and all $\mathcal{A}$-normal structure $w \in \mathfrak{W}$:*

$$\mathcal{K}, w \models \Box\alpha$$

**Proof.** We obtain directly through the results of the theorem 2.7 and the definition of truth in a frame (Def. 2.4 - 5.b, p. 8). ■

## 2.1 Properties of the Accessibility Relation

As we saw, $\mathfrak{R}$ is an acessibility relation defined as $\mathfrak{R}' \cup \mathfrak{R}''$, that:

$$\mathfrak{R}' = \{\langle \mathcal{A}, w\rangle : w \in \mathfrak{W}\}$$

$$\mathfrak{R}'' = \{\langle w, w_1\rangle : w, w_1 \in \mathfrak{W}\}$$

We can get different modal systems according to different imposed conditions over their corresponding relations of accessibility. Let us see some of the properties that $\mathfrak{R}'$ and $\mathfrak{R}''$ can preserve.

**Definition 2.9 ($\mathfrak{R}'$-partial)** *Some $\mathcal{A}$-normal structure is not $\mathfrak{R}'$-related with the partial structure $\mathcal{A}$.*

$$\exists w \in \mathfrak{W} \neg(\mathcal{A}\mathfrak{R}'w)$$

**Definition 2.10 ($\mathfrak{R}'$-total)** *Every $\mathcal{A}$-normal structure is $\mathfrak{R}'$-related with the partial structure $\mathcal{A}$.*

$$\forall w \in \mathfrak{W}(\mathcal{A}\mathfrak{R}'w)$$

**Definition 2.11 ($\mathfrak{R}''$-reflexivity)** *Every $\mathcal{A}$-normal structure is $\mathfrak{R}''$-related with itself.*

$$\forall w \in \mathfrak{W}(w\mathfrak{R}''w)$$

**Definition 2.12 ($\mathfrak{R}''$-simetric)** *Let $w$ and $w_1$ be $\mathcal{A}$-normal structures, if $w$ is $\mathfrak{R}''$-related with $w_1$, then $w_1$ is $\mathfrak{R}''$-related with $w$.*

$$\forall w, w_1 \in \mathfrak{W}(w\mathfrak{R}''w_1 \to w_1\mathfrak{R}''w)$$

**Definition 2.13 ($\mathfrak{R}''$-transitivity)** *Let $w$, $w_1$ and $w_2$ be $\mathcal{A}$-normal structures, if $w$ is $\mathfrak{R}''$-related with $w_1$ and $w_1$ is $\mathfrak{R}''$-related with $w_2$, then $w$ is $\mathfrak{R}''$-related with $w_2$.*

$$\forall w, w_1, w_2 \in \mathfrak{W}(w\mathfrak{R}''w_1 \wedge w_1\mathfrak{R}''w_2 \to w\mathfrak{R}''w_2)$$

**Definition 2.14 ($\mathfrak{R}''$-serial)** *For all $w \in \mathfrak{W}$, there is a $w_i \in \mathfrak{W}$ that $w$ is $\mathfrak{R}''$-related with $w_i$.*

$$\forall w \in \mathfrak{W} \exists w_1 \in \mathfrak{W}(w\mathfrak{R}''w_i)$$

**Definition 2.15 ($\mathfrak{R}''$-euclidean)** *Let $w$, $w_1$ and $w_2$ $\mathcal{A}$-normal structures, if $w$ is $\mathfrak{R}''$-related with $w_1$ and $\mathfrak{R}''$-related with $w_2$, then $w_1$ is $\mathfrak{R}''$-related with $w_2$.*

$$\forall w, w_1, w_2 \in \mathfrak{W}(w\mathfrak{R}''w_1 \wedge w\mathfrak{R}''w_2 \rightarrow w_1\mathfrak{R}''w_2)$$

As we said, given a frame $\mathcal{K}$, the imposed conditions over $\mathfrak{R}$ (given the previous definition), we can obtain different modal systems to $\mathfrak{T}_m$. For example, we can obtain a K system if $\mathfrak{R}$ is just $\mathfrak{R}'$-total. On the other hand, we can obtain S5 system if $\mathfrak{R}$ is $\mathfrak{R}'$-total, $\mathfrak{R}''$-reflective and $\mathfrak{R}''$-euclidean.

**Example 2.16** *: Let $\mathfrak{R}$ be a accessibility relation $\mathfrak{R}'$-partial and $\mathfrak{R}''$-transitivity, we can obtain the following results (shown here in diagrams):*

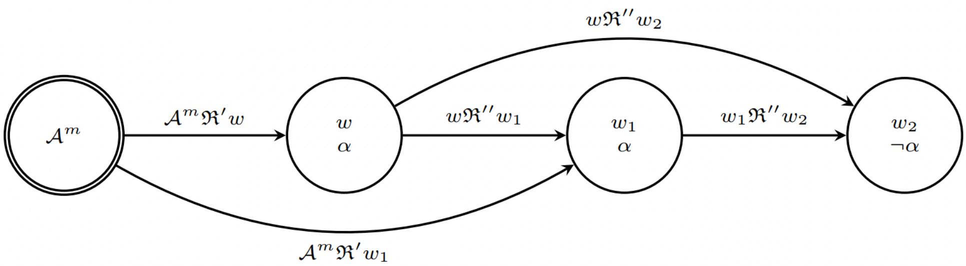


Figure 1: $\mathfrak{R}'$-partial e $\mathfrak{R}''$-transitivity

*Notice that $\mathcal{A}^m$ is not $\mathfrak{R}'$-related with $w_2$ (satisfying $\mathfrak{R}'$-partiality) and, once that $w\mathfrak{R}''w_1$ and $w_1\mathfrak{R}''w_2$, $w$ is $\mathfrak{R}$-related with $w_2$ (satisfying $\mathfrak{R}''$-transitivity). Given the previous structure, follows as theorems:*

$$\mathcal{K}, \mathcal{A}^m\mathfrak{R}w \models \Box\alpha$$

$$\mathcal{K}, \mathcal{A}^m\mathfrak{R}w \models \Diamond\Diamond\neg\alpha$$

$$\mathcal{K}, \mathcal{A}^m\mathfrak{R}w_1 \models \Diamond\Box\neg\alpha$$

## 3 Philosophical and Formal Consequences

For sake of brevity, we will point out in this section some philosophical and formal consequences arising from the possible worlds semantics on the Quasi-Truth theory. We note that here we will only indicate such consequences, which should be examined more carefully in further analysis.

### 3.1 Barcan Formula

As previously mentioned (footnote 12, p. 7), we can define *frames* (Def. 2.1, p. 7) through an alternative approach, in which $\mathfrak{D}$ is defined as a domain function – which associates, for each $w \in \mathfrak{W}$, some non-empty set $\mathfrak{D}(w)$ that can be called "domain of $w$" –, and not just a non-empty set. The approach we have followed so far is called "frames with constant domain", because every $\mathcal{A}^m$-Normal structure $w$, the domain of $w$ is $\mathfrak{D}$. On the other hand, in the alternative approach (called "frames with variable domain"), it could be the case that, given two $\mathcal{A}^m$-Normal structures $w_1$ and $w_2$, the domain $\mathfrak{D}(w_1)$ was different from the domain $\mathfrak{D}(w_2)$. That is, the range of the quantifiers would be restricted to each $w \in \mathfrak{W}$.

However, we should notice that, given the definition of $\mathcal{A}^m$-Normal structures (Def. 1.5, p. 5), every $\mathcal{A}^m$-Normal structure has its domain equal to the modal pragmatic structure $\mathcal{A}^m$. Thus, due to this feature we choose to define a *frame* with constant domain – and so we use the most convenient definition. However, there was nothing to prevent the definition of *frames* through the alternative approach, that is, *frames* with variable domains, imposing the following condition: for every $w_1, w_2 \in \mathfrak{W}$, $\mathfrak{D}(w_1) = \mathfrak{D}(w_2) = D \in \mathcal{A}$ – that is, the domains of the $\mathcal{A}^m$-Normal structures are equals and, in this way, are equals to the domain of the modal pragmatic structure $\mathcal{A}^m$. The previous condition would make our *frame* with constant domain again, but we would only be using a more extensive notation.

This alternative approach to *frames* (and their notation) may, nevertheless, allow us to easily notice two important properties that many structures can preserve, namely, the *monotonicity* and the *anti-monotonicity*. Let us see how these properties are defined.

**Definition 3.1 (Monotonic Structure)** *A structure* $\langle \mathfrak{W}, \mathfrak{R}, \mathfrak{D} \rangle$ *is monotonic if, and only if, for every* $w_1, w_2 \in \mathfrak{W}$*, if* $w_1 \mathfrak{R} w_2$*, then* $\mathfrak{D}(w_1) \subseteq \mathfrak{D}(w_2)$*. A frame is monotonic if its structure is.*

**Definition 3.2 (Anti-Monotonic Structure)** *A structure* $\langle \mathfrak{W}, \mathfrak{R}, \mathfrak{D} \rangle$ *is anti-monotonic if, and only if, for every* $w_1, w_2 \in \mathfrak{W}$*, if* $w_1 \mathfrak{R} w_2$*, then* $\mathfrak{D}(w_2) \subseteq \mathfrak{D}(w_1)$*. A frame is anti-monotonic if its structure is.*

**Lemma 3.3** *If a frame has constant domain, then it is monotonic and anti-monotonic*

**Proof.** Let $\mathcal{K} = \langle \mathcal{A}, \mathfrak{W}, \mathfrak{D}, \mathfrak{R}, \ell \rangle$ be a *frame* with constant domain. Then for every $w_1, w_2 \in \mathfrak{W}$, $\mathfrak{D}(w_1) = \mathfrak{D}(w_2)$. If follows as theorem of the *Zermelo-Fraenkel* set theory that, if $X$ is a set, then $X \subseteq X$ – that is, every set is a subset of itself. Thus, since $\mathfrak{D}(w_1) \subseteq \mathfrak{D}(w_1)$ and $\mathfrak{D}(w_1) = \mathfrak{D}(w_2)$, then by the substitutivity of identicals, $\mathfrak{D}(w_1) \subseteq \mathfrak{D}(w_2)$ (and vice versa). Therefore, for every, $w_1, w_2 \in \mathfrak{W}$, $\mathfrak{D}(w_1) \subseteq \mathfrak{D}(w_2)$ – which characterizes the *monotonicity* – as also $\mathfrak{D}(w_2) \subseteq \mathfrak{D}(w_1)$ – which characterizes the *anti-monotonicity.* ■

These two properties are related with the so-called "Barcan Formula" – and its converse. The Barcan Formula was first obtained by the philosopher and logician *Ruth Barcan Marcus* [1]. Appropriately, both the Barcan Formula and its converse are not exactly formulas, but rather *scheme of formulas*, which we can define as follows.

**Definition 3.4 (Barcan Formula)** *Every formula which has the following form are Barcan Formula:*

$$\Diamond\exists x\varphi \rightarrow \exists x\Diamond\varphi$$

$$\forall x\Box\varphi \rightarrow \Box\forall x\varphi$$

**Definition 3.5 (Converse Barcan Formula)** *Every formula which has the following form are Converse Barcan Formula:*

$$\exists x\Diamond\varphi \rightarrow \Diamond\exists x\varphi$$

$$\Box\forall x\varphi \rightarrow \forall x\Box\varphi$$

An intuitive interpretation of these formulas would be that Barcan Formula determines that nothing comes into existence or moves from one possible world to another; whereas Converse Barcan Formula determines that nothing ceases to exist.

**Theorem 3.6** *If a frame is monotonic, then preserves Converse Barcan Formula.*

**Proof.** Found in [7, pp. 111-112, Prop. 4.9.6]. ■

**Theorem 3.7** *If a frame is anti-monotonic, then preserves Barcan Formula.*

**Proof.** Found in [7, p. 112, Prop. 4.9.8]. ■

Given the previous theorems, it follows that the *frame* $\mathcal{K}$ preserves both Barcan Formula as well as its converse, since it is monotonic and anti-monotonic. This result is important for the intended interpretation of $\mathfrak{I}_m$. If, given $\mathfrak{I}_m$, we obtain that there is an $\mathcal{A}^m$-Normal strucute $w$, such that $w$ is accessible by $\mathcal{A}^m$, and in $w$ there is a $x$ that satisfies a certain sentence $F$, follows that it exists in the structure $\mathcal{A}^m$ an element $x$ that, in $w$, $x$ satisfies $F$. The philosophical result we can get from Barcan Formula is that, given a scientific theory $\mathfrak{T}$, this theory assumes that everything that can exists, exists – or, in other words, everything that can exist is already effectively postulated by the theory. This feature may point the following problem to Quasi-Truth theory.

### 3.1.1 Cardinality and Anti-Monotonicity

The Quasi-Truth theory describe the partiality of a scientific theory trough the partial structures. However, as is clear from the modal extension of $\mathfrak{T}$, here presented, theories must determine their cardinality so that the domain of entities treated by $\mathfrak{T}$ should be *the same* in both pragmatic modal structure as the total structures that extend it. This ensure, as we have seen, the monotonicity and anti-monotonicity of *frame* $\mathcal{K}$. But it seems plausible to ask: why every $\mathcal{A}^m$-Normal structure (which is a total structure), that extend the modal pragmatic structure $\mathcal{A}^m$ (which is a partial structure), should have *the same* domain? At first sight, this condition is imposed so that all elements and relations already defined in the partial structure can also be preserved in the total structure.

Why we do not change the definition of $\mathcal{A}^m$-Normal structure so that the domain of the modal pragmatic structure $\mathcal{A}^m$ is a subset of the domains of the total structures that extend it (instead of being *the same*)? That is:

**Definition 3.8 (Modified $\mathcal{A}^m$-Normal Structures)** *Let $\mathfrak{T}$ be an axiomatized theory in a first-order language, $\mathcal{A}^m = \langle D, R_i, G, Q\rangle_{i\in I}$ a modal pragmatic structure and $\mathcal{B} = \langle D', R'_i, \rho\rangle_{i\in I}$ a total structure, where $\mathfrak{T}$ is interpreted. $\mathcal{B}$ will be an $\mathcal{A}^m$-normal structure if the follow properties are satisfied:*

*(a)* $D \subseteq D'$

*(b) The total relations $(R'_i)_{i\in I}$ of $\mathcal{B}$ extend the corresponding partial relations $(R_i)_{i\in I}$ of $\mathcal{A}^m$.*

*(c) If c is an individual constant of $\mathfrak{T}$, then c is interpreted in $\mathcal{A}^m$ and $\mathcal{B}$ by the same element.*

*(d) If $\alpha \in G$, then $\mathcal{B}$ satisfies $\alpha$. That is, every sentence that belong to G is valid in the structure $\mathcal{B}$, what we denote by:*

$$B \models \alpha.$$

Trough this modification, the domains of the $\mathcal{A}^m$-Normal structures may or may not be different from the modal pragmatic structure $\mathcal{A}^m$. In this way, we can obtain that the *frame* $\mathcal{K}$ will not have constant domain, which follows that it will not preserve the anti-monotonicity – Since there may be $\mathcal{A}^m$-Normal structures of domains with higher cardinality than the $\mathcal{A}^m$ modal pragmatic structure.[13]

The following formal consequences of the suggested modification is that Barcan Formula will not be valid in $\mathcal{K}$. What would be the philosophical consequences? This is a matter for future investigations.

## 4 Further Discussions

Two other topics, to be dealt with in the future, are the existence of necessary sentences known only *a posteriori*, as well as the development of a counterfactual logic through the semantics here presented. That is, can there be some $\alpha \in Q$ (which are sentences whose truth is only empirically decidable) such that $\forall w \in W\ \mathcal{K}, w \models \alpha$? In other words, can there be truths only knowable *a posteriori* that are *necessarily true* according to $\mathfrak{T}_m$?

Regarding so-called "Counterfactual Conditionals", they are distinguished from other forms of conditionals, such as the so-called "material conditional". The material conditional, in the form $P$ *implies* $Q$, has as truth value (in classical logic), which will always be true when its antecedent is false. As well known in the literature, such conditionals allow the so-called "material conditional paradoxes". In counterfactual conditionals, however, the consequent is obtained if the antecedent is an accurate description of reality. The counterfactual conditionals can be seen, for example, in sentences such as *if Socrates were immortal, then he would not be human* or *if the earth had a mass $n$ times smaller, the moon would escape its orbit.* Nevertheless, note that the antecedents can not be effectively evaluated, since Socrates is indeed mortal as the Sun does not have a mass $n$ times smaller than they actually have. Therefore, it is necessary to have a modal semantics to deal with these conditionals, thus being able to evaluate such conditionals in circumstances whose antecedent would be true. What would a semantics for counterfactual conditionals, built on the semantics of possible worlds presented here, be like?

[13]In private conversations with Newton da Costa, he also suggested investigating cases in which the domains of $\mathcal{A}^m$-normal structures are smaller than the domain of the modal pragmatic structure $\mathcal{A}^m$. Another suggestion for further investigations, also made by da Costa, is about the use of Temporal Logic to work with scientific theories and their possible application in the theory here presented.

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