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Principal and Boolean congruences on M_3-lattices

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Abstract

In 2018, A. V. Figallo and M. A. Jiménez presented a Priestley-style topological duality type for the M_3-lattices (see [4]). In this work we continue with the study and describe the principal and Boolean congruences, through the open and closed of the associated Priestley space. Among other results, we prove that both coincide and are associated with ideals generated by the Boolean elements of the algebra, being able to precisely determine what these elements are.

Keywords: M_3-lattices, principal congruences, Boolean congruences, Priestley spaces.

Introduction

The class of the M_3-lattices was defined by Figallo, at the suggestion of A. Monteiro, in [1] and his consideration was motivated by the implementation of certain trivalent switching circuits.

In the aforementioned work, with the aim of finding the simple algebras and proving that the variety was semisimple, Figallo introduced the notion of n-ideal (prime) of an M_3-lattice L, as an ideal (prime) N of L that verifies: if $x \in N$, then $\sim x \in N$, or equivalently, $x \in N$ implies $\nabla x \in N$.

In a later paper ([3]), A. V. Figallo defined: (I) $x|y = x \wedge \triangle(\sim (x \vee \nabla y) \vee \sim (y \vee \sim x))$ and proved that if $\langle L, \wedge, \vee, \sim, \triangle, 0\rangle$ is an M_3-lattice with greatest element 1, then $\langle L, \wedge, \vee, 0, 1\rangle$ is a Brouwer algebra.

This fact allowed him to characterize the congruences using operation (I), and the notion of n-ideal that he had introduced. Demonstrating that the variety is semisimple and that the only simple algebra is $\langle \mathrm{T}, \wedge, \vee, \sim, \triangle, 0\rangle$, where T is the chain with three elements $\{0, 1/2, 1\}$ with $0 \leq 1/2 \leq 1$ and the operations $\sim$ and $\triangle$ are defined in the following table:

x	$\sim x$	$\triangle x$
0	0	0
1/2	1	0
1	1/2	1

He also proved that $\langle \mathrm{T}, \wedge, \vee, -, \nabla, 0, 1\rangle$ is a trivalent Łukasiewicz algebra in the sense of [5], where operations ∇ and $-$ are defined as follows:

(i) $\nabla x = \neg\neg x$,

(ii) $-x = (\neg\neg x \to x) \wedge (x \to \neg x)$.

being

(iii) $x \to y = \triangle{\sim}(x \vee {\sim}1) \vee y$,

(iv) $\neg x = \triangle{\sim}(\nabla x \vee {\sim}1)$

From this result, taking into account a theorem of representation that Figallo had demonstrated for these algebras, he was able to assure that if ∇ and $-$ are indicated in (i) and (ii) respectively, and $\langle L, \wedge, \vee, \sim, \triangle, 0\rangle$ is an M_3-lattice with last element 1, then $\langle L, \wedge, \vee, -, \nabla, 0, 1\rangle$ is centered trivalent Łukasiewicz algebra, with center $c = {\sim}1$.

In this paper we provide a description of the principal congruences in the bounded M_3-lattices, based on the characterization of the congruency lattice we had previously obtained, via Priestly topological duality type, for this class of algebras (see [4]).

This article has been organized as follows. In Section 2 we introduce the definition and properties of M_3-lattices given by Figallo and we briefly describe the duality previosly obtained for bounded M_3-lattices. In Section 3, we show another characterization of the M_3-congruency lattice in terms of the open subsets of their associated topological space. In Sections 4 and 5, applying the results obtained in the previous section, we describe the principal and Boolean congruences, through the open and closed of the associated Priestley space. Among other results, we prove that both coincide and are associated to ideals generated by the Boolean elements of algebra, being able to precisely determine what those elements are.

1 Preliminaries

Let's remember that an M_3-lattice is an algebra $\langle L, \wedge, \vee, \sim, \triangle, 0\rangle$ type $(2, 2, 1, 1, 0)$ such that the reduct $\langle L, \wedge, \vee, 0\rangle$ is a distributive lattice with least element 0 and it satisfies the following identities:

(M1) $\triangle(x \wedge {\sim}x) = 0$,

(M2) $\sim\sim x = x$,

(M3) $x = \triangle x \vee \sim\nabla x$, where $\nabla x = x \vee \sim x$,

(M4) $\sim\triangle x \vee \triangle x = \triangle x$,

(M5) $\triangle\nabla x = \nabla x$,

(M6) $\triangle(x \vee y) = \triangle x \vee \triangle y$,

(M7) $\nabla(x \wedge y) = \nabla x \wedge \nabla y$.

If $\langle L, \wedge, \vee, \sim, \triangle, 0\rangle$ is an M_3-lattice such that the reduct $\langle L, \wedge, \vee\rangle$ is a distributive lattice with greatest element, we say that is a bounded M_3-lattice and we denote with $\mathbf{M_3}$ the variety of the bounded M_3-lattices.

In [6] we present an extension of Priestley duality for the bounded distributive lattices, in the case of M_3-lattices. To do this we introduce the category $\mathfrak{M}_\mathbf{3}$ of M_3-spaces and M_3-functions, where a M_3-space is a triple $(X, \tau, \leq)$ such that:

(MP1) $(X, \tau, \leq)$ is a Priestley space.

(MP2) $(X, \leq)$ is the cardinal sum of a family of chains, each of which has exactly two elements,

(MP3) for each $U \in D(X)$ (where $D(X)$ denote the bounded distributive lattice of the open, closed and decreasing subsets of X) the following properties are verified:

(a) $(M_X U]$ is an open and closed subset in X,

(b) $[m_X U)\setminus M_X U$ is an open and closed subset in X, where $M_X U = max\, X \cap U$, $m_X U = min\, X \cap U$ and $max\, X$ $(min\, X)$ denote the set of maximal (minimal) elements of X.

On the other hand, an M_3-function of an M_3-space $(X, \tau, \leq)$ in an M_3-space $(X', \tau', \leq')$, is a increasing continuous function $h : X \longrightarrow X'$ such that for all $V \in D(X')$ are verified:

(MF1) $(M_X h^{-1}(V)] = h^{-1}((M_{X'} V])$,

(MF2) $[m_X h^{-1}(V)) \setminus M_X h^{-1}(V) = h^{-1}([m_{X'} V) \setminus M_{X'} V)$.

Then we proved:

(DP1) If $\langle L, \wedge, \vee, \triangle, \sim, 0, 1\rangle$ is an M_3-lattice with last element, then the set $\mathcal{I}_p(L)$, of the primes ideals of L, ordered by the inclusion relation and endowed with the topology τ having as a subbasis the sets

(A1) $\sigma_L(a) = \{I \in \mathcal{I}_p(L) : a \notin I\}$ and $\mathcal{I}_p(L) \setminus \sigma_L(a)$, for each $a \in L$,

is an M_3-space, called the M_3-space associated with L. Also the application $\sigma_L : L \longrightarrow D(\mathcal{I}_p(L))$ defined as in (A1), is an M_3-isomorphism.

(DP2) If $(X, \tau, \leq)$ is an M_3-space, and for each $U \subseteq X$, we define:

(D) $\triangle^* U = (M_X U]$,
(N) $\neg U = [m_X U) \setminus M_X U$,
(B) $\nabla^* U = U \cup \neg U$,

then $\langle D(X), \cap, \cup, \triangle^*, \neg, \emptyset, X\rangle$ is a bounded M_3-lattice.

Then we demonstrate, in the usual way, that the category $\mathfrak{M}_\mathbf{3}$ is dually equivalent to the category $\boldsymbol{\mathcal{M}_3}$ of M_3-lattices and M_3-homorphisms.

One of the important facts of Priestley duality is that if L is a bounded distributive lattice, there is a biunivocal correspondence between the congruences of L and the closed subsets of $\mathcal{I}_p(L)$, more precisely H. A. Priestley ([6], [7], [8]) proved that if Y is a closed subset of $\mathcal{I}_p(L)$, then

(A3) $\Theta(Y) = \{(a, b) \in L \times L : \sigma_L(a) \cap Y = \sigma_L(b) \cap Y\}$,

is a congruence over L. Conversely, if θ is a congruence of L and $q : L \longrightarrow L/\theta$ is the canonical epimorphism, then

(A4) $Y = \{q^{-1}(I) : I \in \mathcal{I}_p(L/\theta)\}$,

is a closed subset of $\mathcal{I}_p(L)$ such that $\Theta(Y) = \theta$ and the correspondence $Y \longrightarrow \Theta(Y)$, establishes an isomorphism between $C(\mathcal{I}_p(L))$, the lattice of the closed subsets of L, and the dual of the lattice $Con(L)$ of the congruences on L.

The notion of $\triangle-$involutive set of an M_3-space associated X, as subsets Y of X such that $\triangle^* Y = Y$, allowed us to characterize the lattice of $\mathbf{M_3}-$congruences as follows:

Theorem 1.1 *Let $L \in \mathbf{M_3}$ and $\mathcal{I}_p(L)$ be the M_3-space associated to L. Then the lattice $C_\triangle(\mathcal{I}_p(L))$ of the all closed and $\triangle-$involutive subsets of $\mathcal{I}_p(L)$, is isomorphic to the dual of the lattice $Con_{\mathbf{M_3}}(L)$ of $\mathbf{M_3}-$congruences, and the isomorphism is established by the function $\Theta_{C\triangle}$ defined by the same prescription as that given in* (A3).

It is worth mentioning here that the $\triangle-$involutive sets also admit the following characterization:

(A5) *Let X be an M_3-space and Y be a non-empty subset of X. Then the following conditions are equivalent:*

(i) *Y is $\triangle-$involutive subset,*

(ii) *Y is increasing and decreasing subset,*

(iii) *Y is a cardinal sum of a family of chains, each of which has exactly two elements.*

Before the end of the section we will set some notations necessary for the following. If $\boldsymbol{K}$ is a class of algebra and $A \in \boldsymbol{K}$, we indicate with $Con_{\boldsymbol{K}}(A)$ the set of congruences on A, or also $\boldsymbol{K}-$congruences in order to highlight the class of algebras we are considering. In case this is not necessary we will simply write $Con(A)$. Also, in general, if $a \in A$ and θ is a congruence on A, with $|a|_\theta$, we denote the equivalence class of a. Also if $a, b \in A$, with $\Theta(a, b)$ we denote the principal congruence generated by (a, b), that is, the least congruence such that a and b are in the same equivalence class.

On the other hand, we designate with $\mathbf{L}$, the class of the bounded distributive lattices (or $(0,1)-$ distributive lattices) and we denote by $\mathcal{P}(X)$ the family of subsets of a set X.

2 Another characterization of the M_3-congruences lattice

The duality described in Section 1 allowed us to characterize the lattices of the congruences of an M_3-lattice, in term of certain closed subsets of its associated M_3-space, more precisely the closed and $\triangle-$involutive subsets.

Below we prove that this can also be done with open and $\triangle-$involutive subsets of the associated with an M_3-space.

Lemma 2.1 *Let $L \in \mathbf{M_3}$ and $\mathcal{I}_p(L)$ be the M_3-space associated with L, then for all $U, V \in \mathcal{P}(\mathcal{I}_p(L))$, $\triangle^*(U \setminus V) = \triangle^* U \setminus \triangle^* V$.*

Proof. Let $x \in \triangle^*(U \setminus V)$, then there is $y \in max\, \mathcal{I}_p(L) \cap (U \setminus V)$, such that $x \leq y$. Then it is clear that there is $y \in max\, \mathcal{I}_p(L) \cap U$ such that $x \leq y$ and consequently $x \in \triangle^* U$. If $x \in \triangle^* V$, we would have that there is $z \in max\, \mathcal{I}_p(L) \cap V$ such that $x \leq z$, from where analyzing the different cases that arise: (a) $x < y$ and $x < z$, (b) $x < y$ and $x = z$, (c) $x = y$ and $x < z$, (d) $x = y$ and $x = z$, we would arrive to contradictions. So $x \notin \triangle^* V$ and thus $x \in \triangle^* U \setminus \triangle^* V$.

For the other inclusion, let's consider (1) $x \in \triangle^* U \setminus \triangle^* V$. Then there's $y \in max\,\mathcal{I}_p(L) \cap U$ such that $x \leq y$. If $y \in V$, it would verify that $y \in max\,\mathcal{I}_p(L) \cap V$, which would imply that $x \in \triangle^* V$ that contradicts (1). Then $y \in (U \setminus V) \cap max\,\mathcal{I}_p(L)$, and therefore $x \in \triangle^*(U \setminus V)$ holds. ■

Corollary 2.2 *Let $L \in \mathbf{M_3}$ and $\mathcal{I}_p(L)$ be the M_3-space associated with L. Then, G is an open and $\triangle-$involutive subset of $\mathcal{I}_p(L)$, if and only if $\mathcal{I}_p(L) \setminus G$ is a closed and $\triangle-$involutive subset of $\mathcal{I}_p(L)$.*

Proof. Let G be an open and $\triangle-$involutive subset of $\mathcal{I}_p(L)$, then $\mathcal{I}_p(L) \setminus G$ is a closed subset of $\mathcal{I}_p(L)$. Furthermore, by Lemma 2.1, we have that $\triangle^*(\mathcal{I}_p(L) \setminus G) = (\triangle^* \mathcal{I}_p(L)) \setminus (\triangle^* G)$ and since $\mathcal{I}_p(L)$ and G are $\triangle-$involutive sets, we get $\triangle^*(\mathcal{I}_p(L) \setminus G) = \mathcal{I}_p(L) \setminus G$, from which we conclude that $\mathcal{I}_p(L) \setminus G$ is $\triangle-$involutive subset of $\mathcal{I}_p(L)$. The reciprocal is analogous. ■

Lemma 2.3 *Let $L \in \mathbf{M_3}$ and $\mathcal{I}_p(L)$ be the M_3-space associated with L. If Y is a closed subset of $\mathcal{I}_p(L)$ and $a, b \in L$, the following conditions are equivalent:*

(i) $\sigma_L(a) \cap Y = \sigma_L(b) \cap Y$,

(ii) $(\sigma_L(b) \Delta\, \sigma_L(a)) \cap Y = \emptyset$,

(iii) $(\sigma_L(b) \Delta\, \sigma_L(a)) \subseteq \mathcal{I}_p(L) \setminus Y$, *where $\sigma_L(b)\Delta, \sigma_L(a)$ is the symmetrical difference of $\sigma_L(b)$ with $\sigma_L(a)$.*

Proof. It is routine. ■

Theorem 2.4 *Let $L \in \mathbf{M_3}$ and $\mathcal{I}_p(L)$ be the M_3-space associated with L. Then the lattice $O_\triangle(\mathcal{I}_p(L))$ of open and $\triangle-$involutive subsets of $\mathcal{I}_p(L)$ is isomorphic to the lattice $Con_{\mathbf{M_3}}(L)$ of $\mathbf{M_3}-$congruences on L, and the isomorphism is established by the function $\Theta_{O\triangle}$ defined by:*

(A3') $\Theta_{O\triangle}(G) = \{(a, b) \in L \times L : (\sigma_L(b) \Delta\, \sigma_L(a)) \subseteq G\}$.

Proof. It is a immediate consequence of Lemma 2.3, Corollary 2.2 and Theorem 1.1. ■

3 Principal $\mathbf{M_3}-$congruences

In this section we characterize closed and $\triangle-$involutive subsets and open and $\triangle-$involutive subsets of M_3-space associated with an M_3-lattice, which correspond to principal $\mathbf{M_3}-$congruences under the duality.

With this purpose, we will start by introducing the following results that are necessary for the determination of the principal congruences.

Proposition 3.1 *Let $L \in \mathbf{M_3}$ and let $\mathcal{I}_p(L)$ be the Priestley space associated with L. Let $I \subseteq L$ be an ideal and $\sigma(I) = \{I_p \in \mathcal{I}_p(L) : I \subseteq I_p\}$. Then the following properties are fulfilled:*

(i) *$\sigma(I)$ is closed and increasing subset of $\mathcal{I}_p(L)$.*

(ii) *$\theta(I) = \Theta(\sigma(I))$, where $\theta(I) = \{(a, b) \in L^2 :$ there exists $i \in I$ such that $a \vee i = b \vee i\}$ is the congruence associated with the ideal I, and $\Theta(\sigma(I))$, the congruence defined as in* (A3).

Proof. (i): It is easy to see that $\sigma(I)$ is an increasing set in $\mathcal{I}p(L)$. On the other hand, if $I'_p \in \mathcal{I}_p(L) \setminus \sigma(I)$, there is $a \in L$ such that (1) $a \in I$ and $a \notin I'_p$, therefore $I'_p \in \sigma_L(a)$, it also verifies that $\sigma_L(a) \subseteq \mathcal{I}_p(L) \setminus \sigma(I)$. In fact, if $I''_p \in \sigma_L(a)$, then $a \notin I''_p$ and from (1), $I''_p \notin \sigma(I)$. Therefore $\sigma(I)$ is a closed subset of $\mathcal{I}_p(L)$.

(ii): Let us consider $I \subset L$ an ideal and $a, b \in L$ such that $(a, b) \in \theta(I)$. Then there is (1) $i \in I$ such that (2) $a \vee i = b \vee i$. Let $I_p \in \sigma(I) \cap \sigma_L(a)$, then (3) $I \subseteq I_p$ and $a \notin I_p$, of this last we see that $a \vee i \notin I_p$. Taking into account (2), it is verified that $b \vee i \notin I_p$ from where by (1) and (3) $b \notin I_p$. So that's $I_p \in \sigma(L) \cap \sigma_L(b)$. The other inclusion is proved in an analogous way, therefore $\sigma_L(a) \cap \sigma(I) = \sigma_L(b) \cap \sigma(I)$, and so it is demonstrated that $(a, b) \in \Theta(\sigma(I)$. Then $\theta(I) \subseteq \Theta(\sigma(I))$.

For the other inclusion let us consider $a, b \in L$ such that (4) $(a, b) \in \Theta(\sigma(I))$ and suppose that $(a, b) \notin \theta(I)$. Then for all $i \in I$, $a \vee i \neq b \vee i$ which means that for all $i \in I$, $a \vee i \not\leq b \vee i$, or for all $i \in I$, $b \vee i \not\leq a \vee i$.

Suppose that for all $i \in I$, $a \vee i \not\leq b \vee i$, then for all $i \in I$, $a \not\leq b \vee i$, and then $a \notin I(I \cup \{b\})$, being $I(I \cup \{b\})$ the ideal generated by $I \cup \{b\}$. Then, as a consequence of the Birkhoff-Stone Theorem, there is $I_p \in \mathcal{I}_p(L)$ such that $I(I \cup \{b\}) \subseteq I_p$ and $a \notin I_p$. Therefore $I_p \in \sigma(I) \cap \sigma_L(a)$ and $I_p \notin \sigma_L(b)$, which implies $I_p \notin \sigma_L(b) \cap \sigma(I)$. We have so $\sigma_L(a) \cap \sigma(I) \neq \sigma_L(b) \cap \sigma(I)$, from where $(a, b) \notin \Theta(\sigma(I))$, which contradicts (4). If we assume that for all $i \in I$, $b \vee i \not\leq a \vee i$, we get in an analogous form a similar contradiction, so $(a, b) \in \theta(I)$. ■

Proposition 3.2 *Let $L \in \mathbf{L}$, $\mathcal{I}_p(L)$ be the Priestley space associated with L and $Y \subseteq \mathcal{I}_p(L)$, then the following conditions are equivalent:*

(i) *Y is a closed and increasing subset of $\mathcal{I}_p(L)$,*

(ii) *if $I_p \notin Y$, then there exists $U \in D(\mathcal{I}_p(L))$ such that $I_p \in U$ and $U \cap Y = \emptyset$,*

(iii) *if $I_p \notin Y$, there exists $a \in L$ such that $a \notin I_p$ and $a \in Z_p$ for all $Z_p \in Y$,*

(iv) *there is an ideal I of L such that $\sigma(I) = Y$, and, in addition, $I = \bigcap_{Q \in Y} Q$.*

Proof. (i) $\Rightarrow$ (ii): Let Y be a closed and increasing subset of $\mathcal{I}_p(L)$, such that $I_p \notin Y$. Then as Y is increasing, for all $Z_p \in Y$, $Z_p \nsubseteq I_p$, which implies, because it is a Priestley space, that for each $Z_p \in Y$, there is $U_Z \in D(X)$ such that $Z_p \notin U_Z$ and (1) $I_p \in U_Z$. From the above we have to (2) $Y \subseteq \bigcup_{Z_p \in Y} (\mathcal{I}_p(L) \setminus U_Z)$.
On the other hand since Y is a closed subset of compact space $\mathcal{I}_p(L)$ then Y is compact, and therefore from (2), there are $Z_1, Z_2, \ldots, Z_n$ such that (3) $Y \subseteq \mathcal{I}_p(L) \setminus \bigcap_{i=1}^{n} U_{Z_i}$. So if $U = \bigcap_{i=1}^{n} U_{Z_i}$, by (1) and (3), we have that $I_p \in U$ and $U \cap Y = \emptyset$.

(ii) $\Rightarrow$ (iii): Let $I_p \notin Y$, then for (ii), there is $U = \sigma_L(a) \in D(\mathcal{I}_p(L))$ such that $I_p \in U$ and $U \cap Y = \emptyset$. Then there is $a \in L$ such that $a \notin I_p$ and $a \in Z_p$, for all $Z_p \in Y$.

(iii) $\Rightarrow$ (iv): It is clear that $I = \bigcap_{Q \in Y} Q$ is an ideal of L, such that $Q \in \sigma(I)$ for all $Q \in Y$, which implies that $Y \subseteq \sigma(I)$. In order to prove the other inclusion, let (1) $I_p \notin Y$, then for (iii), there is $a \in L$ such that (2) $a \notin I_p$ and $a \in Q$ for all $Q \in Y$, therefore (3) $a \in I$. From (2) and (3), $I \nsubseteq I_p$, resulting (4) $I_p \notin \sigma(I)$. Finally by (1) and (4) we have to $\mathcal{I}_p \setminus Y \subseteq \mathcal{I}_p(L) \setminus \sigma(I)$, and therefore $\sigma(I) \subseteq Y$.

(iv) $\Rightarrow$ (i): It is verified by part (i) of Proposition 3.1. ■

In order to characterize the principal congruences on an M_3-lattice through duality, we had to use the notion of convex set, whose definition we give below.

Definition 3.3 *Let $(X, \leq)$ be an order set. A subset Y of X is convex, if $x, y \in Y$ and $x \leq z \leq y$, implies $z \in Y$.*

Remark 3.4 *If $L \in \mathbf{M_3}$ and $\mathcal{I}_p(L)$ is its associated M_3-space, then all subset Y of $\mathcal{I}_p(L)$ is convex, for being the space, a cardinal sum of chains which has exactly two elements.*

Proposition 3.5 *If R is an open, closed and convex set in a Priestley space X and $D(X)$ is the set of open, closed and decreasing subsets of X, then there are $U, V \in D(X)$, such that $U \subseteq V$ and $R = V \setminus U$.*

Proof. It is a consequence of the following results, whose demonstration we expose in each case:

(i) $(R] = \{x \in X :$ there is $y \in R$ such that $x \leq y\}$ is a closed set in X:

Let $x \in X \setminus (R]$, then $x \not\leq y$ for all $y \in R$. Since X is disconnected in the order, for each $y \in R$ there is $U_{xy} \in D(X)$ such that $y \in U_{x,y}$ and $x \notin U_{xy}$. Consequently, $R \subseteq \bigcup_{y \in R} U_{x,y}$ and since X is compact, there are $y_1, y_2, \ldots, y_n \in R$, such that $R \subseteq \bigcup_{i=1}^{n} U_{xy_i}$. Let $U = \bigcup_{i=1}^{n} U_{xy_i}$, then $V = X \backslash U$ is an open and increasing set, $x \in V$ and (1) $R \cap V = \emptyset$. Suppose that $(R] \cap V \neq \emptyset$, then there is $z \in X$ such that $z \in (R]$ and $z \in V$. Therefore there is $s \in R$ such that $z \leq s$ and for being V increasing set, $s \in V$. Then $s \in R \cap V$, which contradicts (1), therefore $V \subseteq X \setminus (R]$. Hence, for each $x \in X \setminus (R]$, there is V, open set in X, such that $x \in V$ and $V \subseteq X \setminus (R]$, which implies that $X \setminus (R]$ is an open set in X and so $(R]$ is a closed set in X.

(ii) $(R] \setminus R$, is decreasing:

Let (1) $x \in (R] \setminus R$ and $y \in X$ such that $y \leq x$. Then there is $z \in R$ such that $x \leq z$ and consequently $y \leq x \leq z$ with $z \in R$, which implies that $y \in (R]$. If $y \in R$, $x \in R$ because R is compact, which contradicts (1). So $y \in (R] \setminus R$.

(iii) There is $U \in D(X)$ such that (a) $(R] \setminus R \subseteq U$ and (b) $R \cap U = \emptyset$:

Let $x \in R$ and $y \in (R] \setminus R$. As $(R] \setminus R$ is decreasing we have to $x \not\leq y$. Then for every $y \in (R] \setminus R$ there is $U_{xy} \in D(X)$ such that $y \in U_{xy}$ and $x \notin U_{xy}$, which implies that (1) $(R] \setminus R \subseteq \bigcup_{y \in R} U_{xy}$. For being R and $(R]$ closed set of X, it is verified that (2) $(R] \setminus R$ is compact and therefore of (1) and (2), there are $y_1, y_2, \ldots, y_n \in R$, such that $(R] \setminus R \subseteq \bigcup_{i=1}^{n} U_{xy_i}$. It is clear that for every $x \in R$, there is $U_x = \bigcup_{i=1}^{n} U_{xy_i} \in D(X)$, such that (3) $(R] \setminus R \subseteq U_x$ y $x \in X \setminus U_x$. It turns out that $R \subseteq \bigcup_{x \in R} (X \setminus U_x)$ and as R is compact, there are $x_1, x_2, \ldots, x_m \in R$ such that $R \subseteq \bigcup_{i=1}^{m} (X \setminus U_{x_i})$ or the equivalent (4) $R \subseteq X \setminus (\bigcap_{i=1}^{m} U_{x_i})$. If $U = \bigcap_{i=1}^{m} U_{x_i} \in D(X)$, by (3) and (4), (a) and (b) hold.

(iv) $V = R \cup U \in D(X)$ and $V \setminus U = R$, where U is the set whose existence assures (iii): Since R is an open and closed set, we can say that $V = R \cup U$ is an open and closed subset of X. We will prove that V is decreasing. Let $x \in V$ and $y \leq x$. We distinguish two cases: (a) $x \in U$ or (b) $x \in R$.

If (a) occurs, it is clear that $y \in V$, for being U decreasing. If (b) is verified, we have that $y \in (R]$. Then if $y \in R$ turns out that $y \in V$. If it were $y \notin R$, it is verified that $y \in (R] \setminus R$, from which by (iii), $y \in U$ and therefore $y \in V$, with which it is demonstrated that $V \in D(X)$. Besides, from (b) of (iii), it is immediate that $V \setminus U = R$. ∎

In [3], Figallo proved that if L is an M_3-lattice, the set $K(L) = \{\triangle x : x \in L\}$, of the invariant element of L, is a generalized Boole algebra. In this fact, if L has as its greatest element 1, it is verified that $1 \in K(L)$, and therefore $[0,1] = \{x \in K(L) : 0 \leq x \leq 1\}$, that coincides with $K(L)$, is a Boole algebra, such that if $x \in K(L)$, then $\overline{x} = \triangle{\sim}(x \vee {\sim}1)$ is its Boolean complement.

Proposition 3.6 *Let $L \in \mathbf{M_3}$, $\mathcal{I}_p(L)$ be its M_3-associated space and $\sigma_L : L \longrightarrow D(\mathcal{I}_p(L))$ given as in* (A1) *of the* Section 1*. Then the restriction of the function σ_L to $K(L)$ is a Boolean isomorphism and for all $d \in K(L)$, $\mathcal{I}_p(L) \setminus \sigma_L(d) = \sigma_L(\overline{d})$, where $\overline{d}$ is the Boolean complement in $K(L)$.*

Proof. To prove that the restriction of σ_L to $K(L)$ is a Boolean isomorphism we only need to prove that for all $x \in K(L)$, $\sigma_L(x) \in K(D(\mathcal{I}_p(L))$ and $\sigma_L(\overline{x}) = \overline{\sigma_L(x)}$, where $\overline{\sigma_L(x)}$ is the Boolean complement of $\sigma_L(x)$ in $K(D(\mathcal{I}_p(L))$.

First of all note that if $x \in K(L)$, then $\triangle x = x$ and since the application $\sigma_L : L \longrightarrow D(\mathcal{I}_p(L))$ is an $\mathbf{M_3}-$isomorphism, we have to $\sigma_L(x) = \triangle^* \sigma_L(x)$, resulting in this way that $\sigma_L(x) \in K(D(\mathcal{I}_p(L))$.

Consequently, since the restriction $\sigma_L | K(L) : K(L) \longrightarrow K(D(\mathcal{I}_p(L)))$ is a bounded lattices homomorphism between Boole algebras, then $\sigma_L | K(L)$ also preserves the complement, that is, $\sigma_L(\overline{x}) = \sigma_L(\triangle{\sim}(x \vee {\sim}1)) = \triangle^* \neg(\sigma_L(x) \cup \neg \mathcal{I}_p(L)) = \overline{\sigma_L(x)}$. Therefore $\sigma_L | K(L)$ is a Boolean isomorphism.

Let us see now that $\mathcal{I}_p(L) \setminus \sigma_L(d) = \sigma_L(\overline{d})$, for all $d \in K(L)$. Let $P \in \mathcal{I}_p(L) \setminus \sigma_L(d)$ with $d \in K(L)$. If $\overline{d} \in P$, since $d \in P$, then $d \vee \overline{d} = 1 \in P$. Therefore P would not be an proper ideal, which contradicts that P is a prime ideal. Then $P \in \sigma_L(\overline{d})$. Let us prove the other inclusion. Let $P \in \sigma_L(\overline{d})$, then $\overline{d} \notin P$. Since $0 = d \wedge \overline{d} \in P$, and P is prime ideal, it is verified that $d \in P$. Hence $P \notin \sigma_L(d)$, which is equivalent to $P \in \mathcal{I}_p(L) \setminus \sigma_L(d))$. ■

Remark 3.7 *If L is an M_3-lattice and $\Theta(a,b)$ is a principal congruence, we can assume that $a \leq b$, otherwise we consider $a \wedge b$ and $a \vee b$, because $\Theta(a,b) = \Theta(a \wedge b, a \vee b)$.*

Lemma 3.8 *Let $L \in \mathbf{M_3}$, $\mathcal{I}_p(L)$ be the associated M_3-space with L and $a, b \in L$ such that $a \leq b$. If Y is a closed and $\triangle-$involutive subset of $\mathcal{I}_p(L)$, then the following conditions are equivalent:*

(i) $(a,b) \in \Theta_{C\triangle}(Y)$,

(ii) $(\sigma_L(b) \Delta \, \sigma_L(a)) \cap Y = \emptyset$,

(iii) $(\sigma_L(b) \setminus \sigma_L(a)) \cap Y = \emptyset$,

(iv) $(a,b) \in \Theta_{O\triangle}(\mathcal{I}_p(L) \setminus Y)$.

Proof. It is a direct consequence of Lemma 2.3 and Theorem 2.4, because taking into account that $\sigma_L : L \longrightarrow D(X(L))$ is a lattice isomorphism, if $a \leq b$, we have that $\sigma_L(a) \subseteq \sigma_L(b)$, hence $\sigma_L(b)\Delta\, \sigma_L(a) = \sigma_L(b) \setminus \sigma_L(a)$. ■

Definition 3.9 *Let X be an M_3-space and let $C_\triangle(X)$ be the family of all closed and $\triangle-$involutive subsets of X. We say that $Y \in C_\triangle(X)$ is maximally disjointed with a subset R of X in $C_\triangle(X)$, if $Y \cap R = \emptyset$ and for all $Z \in C_\triangle(X)$ such that $Z \cap R = \emptyset$, is verified that $Z \subseteq Y$.*

Lemma 3.10 *Let $L \in \mathbf{M_3}$ and let $\mathcal{I}_p(L)$ be the M_3-space associated with L. For every $a,b \in L$ such that $a \leq b$, if $Y \in C_\triangle(\mathcal{I}_p(L))$ the following conditions are equivalent:*

(i) $\Theta_{C\triangle}(Y) = \Theta(a,b)$,

(ii) *Y is maximally disjointed with the open and closed set $\sigma_L(b)\setminus\sigma_L(a)$ in $C_\triangle(\mathcal{I}_p(L))$.*

Proof. (i) $\Rightarrow$ (ii): Let $Y \in C_\triangle(\mathcal{I}_p(L))$ such that $\Theta_{C\triangle}(Y) = \Theta(a,b)$ with $a \leq b$ and let $F \in C_\triangle(\mathcal{I}_p(L))$ be disjointed with $\sigma_L(b) \setminus \sigma_L(a)$. From Lemma 3.8, we know that $Y \cap (\sigma_L(b) \setminus \sigma_L(a)) = \emptyset$ and that $(a,b) \in \Theta_{C\triangle}(F)$. Consequently $\Theta_{C\triangle}(Y) \subseteq \Theta_{C\triangle}(F)$ and as $\Theta_{C\triangle}$ is an antiisomorphism, it is verified that $F \subseteq Y$. Thus (ii) holds.

(ii) $\Rightarrow$ (i): We consider that Y is maximal in the family of subsets of $\mathcal{I}_p(L)$ closed, $\triangle-$volutive and disjointed with $\sigma_L(b) \setminus \sigma_L(a)$. Then for Lemma 3.8, $\Theta_{C\triangle}(Y)$ is an $\mathbf{M_3}-$congruence such that $(a,b) \in \Theta_{C\triangle}(Y)$ and therefore $\Theta(a,b) \subseteq \Theta_{C\triangle}(Y)$. Besides, for Theorem 1.1, there is a closed and $\triangle-$involutive subset F of $\mathcal{I}_p(L)$ such that $\Theta(a,b) = \Theta_{C\triangle}(F)$. Therefore we have that $\Theta_{C\triangle}(F) \subseteq \Theta_{C\triangle}(Y)$. From this last assertion, as $\Theta_{C\triangle}$ is an antiisomorphism, it turns out that (1) $Y \subseteq F$. On the other hand as $(a,b) \in \Theta_{C\triangle}(F)$, it is verified that (2) $F \cap (\sigma_L(b) \setminus \sigma_L(a)) = \emptyset$. Hence, by (1), (2) and the maximality of Y, we have that $Y = F$ and consequently $\Theta(a,b) = \Theta_{C\triangle}(Y)$. ■

Proposition 3.11 *Let $L \in \mathbf{M_3}$ and let $\mathcal{I}_p(L)$ be the M_3-space associated with L. If $Y \in C_\triangle(\mathcal{I}_p(L))$, then the following conditions are equivalent:*

(i) *$\Theta_{C\triangle}(Y)$ is a principal $\mathbf{M_3}-$congruence on L,*

(ii) *there is an open and closed subset R of $\mathcal{I}_p(L)$, such that Y is maximally disjointed with R in $C_\triangle(\mathcal{I}_p(L))$.*

Proof. (i) $\Rightarrow$ (ii): It follows from Lemma 3.10, where $R = \sigma_L(b) \setminus \sigma_L(a)$.
(ii) $\Rightarrow$ (i): Let $Y \in C_\triangle(\mathcal{I}_p(L))$ such that there exists $R \subseteq \mathcal{I}_p(L)$ which verifies:

(a) R is an open and closet set,

(b) $Y \cap R = \emptyset$,

(c) if $F \in C_\triangle(\mathcal{I}_p(L))$ and $F \cap R = \emptyset$, then $F \subseteq Y$.

From (a), Proposition 3.5 and Observation 3.4, there are $U, V \in D(\mathcal{I}_p(L))$ such that $U \subseteq V$ and $V \setminus U = R$. Then there are $a, b \in L$ with $a \leq b$ such that $U = \sigma_L(a)$ and $V = \sigma_L(b)$. Besides, by (b) and Lemma 3.8, we obtain (1) $(a, b) \in \Theta_{C\triangle}(Y)$. Let (2) $\vartheta \in Con_{\mathbf{M_3}}(L)$ such that (3) $(a, b) \in \vartheta$. Then, for Theorem 1.1, there is $F \in C_\triangle(\mathcal{I}_p(L))$ such that $\vartheta = \Theta_{C\triangle}(F)$, which implies $(a, b) \in \Theta_{C\triangle}(F)$ and consequently, for Lemma 3.8, $(\sigma_L(b) \setminus \sigma_L(a)) \cap F = \emptyset$. From the latter results, by (c), that $F \subseteq Y$, and since $\Theta_{C\triangle}$ is an antiisomorphism we have that (4) $\Theta_{C\triangle}(Y) \subseteq \Theta_{C\triangle}(F) = \vartheta$. Then, from (1), (2), (3) and (4), we infer that $\Theta_{C\triangle}(Y) = \Theta_{C\triangle}(a, b)$ and therefore it is a principal $\mathbf{M_3}-$congruence. ■

Proposition 3.12 *Let $L \in \mathbf{M_3}$, $\mathcal{I}_p(L)$ be the M_3-space associated with L and $a, b \in L$ such that $a \leq b$. If $G \in O_\triangle(\mathcal{I}_p(L))$, then the following conditions are equivalent:*

(i) $\Theta_{O\triangle}(G) = \Theta(a, b)$,

(ii) *G is least element of $O_\triangle(\mathcal{I}_p(L))$, in the sense of inclusion, which contains $\sigma_L(b) \setminus \sigma_L(a)$.*

Proof. (i) $\Rightarrow$ (ii): Let $G \in O_\triangle(\mathcal{I}_p(L))$ such that $\Theta_{O\triangle}(G) = \Theta(a, b)$. Since $(a, b) \in \Theta_{O\triangle}(G)$, then by Lemma 3.8, it is verified that (1) $(\sigma_L(b) \setminus \sigma_L(a)) \subseteq G$.

On the other hand as $\Theta_{O\triangle}(G) = \Theta_{C\triangle}(\mathcal{I}_p(L) \setminus G)$, then $\Theta_{C\triangle}(\mathcal{I}_p(L) \setminus G) = \Theta(a, b)$ and by Lemma 3.10, $F = \mathcal{I}_p(L) \setminus G$ is maximally disjointed with $\sigma_L(b) \setminus \sigma_L(a)$ in $C_\triangle(\mathcal{I}_p(L))$.

Let $G' \in O_\triangle(\mathcal{I}_p(L))$ such that $\sigma_L(b) \setminus \sigma_L(a) \subseteq G'$, then (2) $(\sigma_L(b) \setminus \sigma_L(a)) \cap (\mathcal{I}_p(L) \setminus G') = \emptyset$ and (3) $F' = \mathcal{I}_p(L) \setminus G' \in C_\triangle(\mathcal{I}_p(L))$.

Hence, from (2) and (3), for the maximality of F, it turns out that $\mathcal{I}_p(L) \setminus G' \subseteq \mathcal{I}_p(L) \setminus G$ and consequently $G \subseteq G'$. Therefore, by (1), G is the least element of subset of $O_\triangle(\mathcal{I}_p(L))$, in the sense of inclusion, which contains $\sigma_L(b) \setminus \sigma_L(a)$.

(ii) $\Rightarrow$ (i): From the hypothesis $(\sigma_L(b) \setminus \sigma_L(a)) \subseteq G$, by Lemma 3.8, we have that $(a, b) \in \Theta_{O\triangle}(G)$. On the other hand, if $\vartheta \in Con_{\mathbf{M_3}}(L)$ is such that $(a, b) \in \vartheta$, then by Theorem 2.4, there is $G' \in O_\triangle(\mathcal{I}_p(L))$ such that $\vartheta = \Theta_{O\triangle}(G')$ and by Lemma 3.8, $\sigma_L(b) \setminus \sigma_L(a) \subseteq G'$.

From the latter, taking into account that G is the least set which contains $\sigma_L(b) \setminus \sigma_L(a)$, we infer that $G \subseteq G'$ and consequently, as $\Theta_{O\triangle}$ is an isomorphism, we have that $\Theta_{O\triangle}(G) \subseteq \Theta_{O\triangle}(G')$.

This way allow us to conclude that $\Theta_{O\triangle}(G)$ is the least $\mathbf{M_3}-$congruence which contains the par (a, b) and therefore $\Theta_{O\triangle}(G) = \Theta(a, b)$. ■

Proposition 3.13 *Let $L \in \mathbf{M_3}$, $\mathcal{I}_p(L)$ be the M_3-space associated with L and $a, b \in L$ such that $a \leq b$. If $G \in O_\triangle(\mathcal{I}_p(L))$, then the following conditions are equivalent:*

(i) $\Theta_{O\triangle}(G) = \Theta(a, b)$,

(ii) $G = \bigcup\limits_{C_i \cap (V \setminus U) \neq \emptyset} C_i$, *where C_i is a maximal chain in $\mathcal{I}_p(L)$, $V = \sigma_L(b)$ and $U = \sigma_L(a)$.*

Proof. (i) $\Rightarrow$ (ii): Let $\Theta_{O\triangle}(G) = \Theta(a, b)$, then by Proposition 3.12, G is the least subset of $O_\triangle(\mathcal{I}_p(L))$, in the sense of inclusion, which contains $\sigma_L(b) \setminus \sigma_L(a)$ and, as G is an $\triangle-$involutive set of $\mathcal{I}_p(L)$, by (A5), $G = \bigcup\limits_{i \in I} C_i$, with C_i maximal chains (chains of two-element), for all $i \in I$.

On the other hand, since $\sigma_L(b) \setminus \sigma_L(a) \subseteq G$, there is a set $I_0 \subseteq I$ such that $C_i \cap (\sigma_L(b) \setminus \sigma_L(a)) \neq \emptyset$, for all $i \in I_0$, and $\bigcup\limits_{i \in I_0} C_i \subseteq G$.

Suppose now, that $\bigcup\limits_{i \in I_0} C_i \subset G$, then there is (1) $P \in G$ and $j \notin I_0$, such that the maximal chain $C_j \subseteq G$ verifies that (2) $P \in C_j$ and $C_j \cap (\sigma_L(b) \setminus \sigma_L(a)) = \emptyset$. As $\mathcal{I}_p(L)$ is a space T_2, all finite set is closed, consequently C_j is closed and also $\triangle-$involutive set.

Then $\mathcal{I}_p(L) \setminus C_j$ is an open and $\triangle-$involutive subset of $\mathcal{I}_p(L)$ such that $\sigma_L(b) \setminus \sigma_L(a) \subseteq \mathcal{I}_p(L) \setminus C_j$. By the minimality of G, we get that $G \subseteq \mathcal{I}_p(L) \setminus C_j$, or the equivalent (3) $C_j \subseteq \mathcal{I}_p(L) \setminus G$. Therefore, from (2) and (3) we conclude that $P \in \mathcal{I}_p(L) \setminus G$, which contradicts (1). So $G = \bigcup\limits_{C_i \cap (V \setminus U) \neq \emptyset} C_i$, where C_i is a maximal chain in $\mathcal{I}_p(L)$, $V = \sigma_L(b)$ and $U = \sigma_L(a)$.

(ii)$\Rightarrow$ (i): Assume that $G = \bigcup\limits_{C_i \cap (U \setminus V) \neq \emptyset} C_i$, with C_i maximal chain in $\mathcal{I}_p(L)$, being $V = \sigma_L(b)$ and $U = \sigma_L(a)$. Let us prove that $V \setminus U \subseteq G$.

Let $P \in V \setminus U$. Since the space is the cardinal sum of chains of two-elements, we infer that $P \in C_P$, being C_P the chain of two-elements that contains to P. Therefore, $P \in C_P \cap (V \setminus U) \subseteq \bigcup\limits_{C_i \cap (V \setminus U) \neq \emptyset} C_i = G$.

Let $G' \in O_\triangle(\mathcal{I}_p(L))$ such that (1) $V \setminus U \subseteq G'$. Then we have that $G \subseteq G'$. Indeed, suppose that $R \in G = \bigcup\limits_{C_i \cap (V \setminus U) \neq \emptyset} C_i$, then there is i_0 such that $R \in C_{i_0}$ and

(2) $C_{i_0} \cap (V \setminus U) \neq \emptyset$. Then, by (1) and (2), $C_{i_0} \cap G' \neq \emptyset$ and, as G' is an $\triangle-$involutive set, we have that $C_{i_0} \subseteq G'$, which implies that $R \in G'$.

We have thus proved that G is the least $\triangle-$involutive set such that $\sigma_L(b) \setminus \sigma_L(a) \subseteq G$. Hence, by Proposición 3.12, we conclude that $\Theta_{O\triangle}(G) = \Theta(a,b)$. ■

Proposition 3.14 *Let $L \in \mathbf{M_3}$, $\mathcal{I}_p(L)$ be the M_3-space associated with L and $a, b \in L$ such that $a \leq b$. If $G \in O_\triangle(\mathcal{I}_p(L))$, then the following conditions are equivalent:*

(i) *$\Theta_{O\triangle}(G)$ is a principal $\mathbf{M_3}-$congruence on L.*

(ii) *there is an open and closed subset R of $\mathcal{I}_p(L)$, such that $G = \bigcup\limits_{C_i \cap R \neq \emptyset} C_i$, with C_i maximal chain in $\mathcal{I}_p(L)$.*

Proof. (i) $\Rightarrow$ (ii): It follows immediately from Proposition 3.13.
(ii) $\Rightarrow$ (i): Let R be a set such as the hypothesis poses. Then Proposition 3.5 assures us that there are $U, V \in D(\mathcal{I}_p(L))$, such that $U \subseteq V$ and $R = V \setminus U$, since, by Observation 3.4, R is a convex set.

On the other hand, since σ_L is an isomorphism, there are $a, b \in L$ such that $a \leq b$, $U = \sigma_L(a)$ and $V = \sigma_L(b)$. Consequently $G = \bigcup\limits_{C_i \cap (V \setminus U) \neq \emptyset} C_i$, with C_i maximal chain in $\mathcal{I}_p(L)$, $V = \sigma_L(b)$ and $U = \sigma_L(a)$.

Then, by Proposition 3.13, we can affirm that $\Theta_{O\triangle}(G) = \Theta(a,b)$, and therefore $\Theta_{O\triangle}(G)$ is a principal $\mathbf{M_3}-$congruence. ■

Proposition 3.15 *Let $L \in \mathbf{M_3}$, $\mathcal{I}_p(L)$ be the M_3-space associated with L and $a, b \in L$ such that $a \leq b$. If $G \in O_\triangle(\mathcal{I}_p(L))$, then the following conditions are equivalent:*

(i) *$\Theta_{O\triangle}(G) = \Theta(a,b)$,*

(ii) *$G = (\nabla^*\sigma_L(b) \setminus \nabla^*\sigma_L(a)) \cup (\triangle^*\sigma_L(b) \setminus \triangle^*\sigma_L(a))$.*

Proof. It results from what was seen in Proposition 3.13 and the fact that (I) $(\nabla^*\sigma_L(b) \setminus \nabla^*\sigma_L(a)) \cup (\triangle^*\sigma_L(b) \setminus \triangle^*\sigma_L(a)) = \bigcup\limits_{C_i \cap (V \setminus U) \neq \emptyset} C_i$, with C_i maximal chain in $\mathcal{I}_p(L)$, $U = \sigma_L(a)$ and $V = \sigma_L(b)$.

The demonstration of (I) is long, for this reason we expose it here.

Let $U = \sigma_L(a)$ and $V = \sigma_L(b)$.

(i) $(\nabla^*\sigma_L(b) \setminus \nabla^*\sigma_L(a)) \cup (\triangle^*\sigma_L(b) \setminus \triangle^*\sigma_L(a)) \subseteq \bigcup\limits_{C_i \cap (V \setminus U) \neq \emptyset} C_i$:

(1) $x \in (\nabla^*\sigma_L(b) \setminus \nabla^*\sigma_L(a)) \cup (\triangle^*\sigma_L(b) \setminus \triangle^*\sigma_L(a))$,

(2) $x \in (\nabla^*\sigma_L(b) \setminus \nabla^*\sigma_L(a))$ or $x \in \triangle^*\sigma_L(b) \setminus \triangle^*\sigma_L(a))$, [(1)]

If

(3) $x \in (\nabla^*\sigma_L(b) \setminus \nabla^*\sigma_L(a))$, [(2)]

(3.1) (a) $x \in \sigma_L(b)$ or (b) $x \in \neg\sigma_L(b)$, [(3), (DP2) (B)]

(3.2) $x \notin \sigma_L(a)$ and $x \notin \neg\sigma_L(a)$. [(3), (DP2) (B)]

If in (3.1) occurs (a), then
(3.1.a.1) $x \in \sigma_L(b) \setminus \sigma_L(a))$, [(3.1)(a), (3.2)]

(3.1.a.2) $x \in C_x$, where C_x is the two-element chain
that contains x, [$\mathcal{I}_p(L)$ is an M_3−space]

(3.1.a.3) $C_x \cap (\sigma_L(b) \setminus \sigma_L(a)) \neq \emptyset$, [(3.1.a.1), (3.1.a.2)]

(3.1.a.4) $x \in C_x \subseteq \bigcup\limits_{C_i \cap (\sigma_L(b) \setminus \sigma_L(a)) \neq \emptyset} C_i$. [(3.1.a.2), (3.1.a.3)]

If in (3.1) occurs (b), then
(3.1.b.1) there is $y \in min\mathcal{I}_p(L) \cap \sigma_L(b)$ such that $y \leq x$, and
$x \notin max\mathcal{I}_p(L) \cap \sigma_L(b)$. [(3.1)(b), (DP2) (N)]

If in (3.1.b.1)
(3.1.b.2) $x \notin max\mathcal{I}_p(L)$,

we have to
(3.1.b.3) $x \in min\mathcal{I}_p(L)$, [(3.1.b.2), $\mathcal{I}_p(L)$ is an M_3−space]

therefore
(3.1.b.4) $y = x$, [((3.1.b.1), (3.1.b.3)]

(3.1.b.5) $x \in \sigma_L(b) \setminus \sigma_L(a)$, [(3.1.b.1), (3.1.b.4), (3.2)]
(3.1.b.6) $x \in C_x$, where C_x is the two-element chain
that contains x, [$\mathcal{I}_p(L)$ is an M_3−space]

(3.1.b.7) $C_x \cap (\sigma_L(b) \setminus \sigma_L(a)) \neq \emptyset$, [(3.1.b.5), (3.1.b.6)]

(3.1.b.8) $x \in C_x \subseteq \bigcup\limits_{C_i \cap (\sigma_L(b) \setminus \sigma_L(a)) \neq \emptyset} C_i$. [(3.1.b.6), (3.1.b.7)]

If in (3.1.b.1)
(3.1.b.9) $x \notin \sigma_L(b)$,

(3.1.b.10) $y < x$, [(3.1.b.1), (3.1.b.9)]

(3.1.b.11) $x \in C_y$, where C_y is the two-element chain
that contains y, [(3.1.b.10), $\mathcal{I}_p(L)$ is an M_3−space]

(3.1.b.12) $y \in \sigma_L(b) \setminus \sigma_L(a)$, [(3.1.b.1), (3.1.b.10), (3.2)]

(3.1.b.13) $C_y \cap (\sigma_L(b) \setminus \sigma_L(a)) \neq \emptyset$, [(3.1.b.11), (3.1.b.12)]

(3.1.b.14) $x \in C_y \subseteq \bigcup_{C_i \cap (\sigma_L(b) \setminus \sigma_L(a)) \neq \emptyset} C_i$. [(3.1.b.11), (3.1.b.13)]

If in (2)

(4) $x \in \triangle^* \sigma_L(b) \setminus \triangle^* \sigma_L(a))$,

(4.1) $x \in (M_{\mathcal{I}_p(L)} \sigma_L(b)]$, [(4), (DP2) (D)]

(4.2) there is $y \in max \mathcal{I}_p(L) \cap \sigma_L(b)$ such that $x \leq y$. [(4.1)]

If

(4.3) $y \in \sigma_L(a)$,

(4.4) $x \in (M_{\mathcal{I}_p(L)} \sigma_L(a)]$, [(4.2), (4.3)]

(4.5) $x \in \triangle^* \sigma_L(a)$, [(4.4), (DP2) (D)]

(4.6) (4.5) contradicts (4),

(4.7) $y \notin \sigma_L(a)$, [(4.3), (4.6)]

(4.8) $y \in \sigma_L(b) \setminus \sigma_L(a)$, [(4.2), (4.7)]

(4.9) $C_y \cap (\sigma_L(b) \setminus \sigma_L(a)) \neq \emptyset$, [(4.8)]

(4.10) $x \in C_y \subseteq \bigcup_{C_i \cap (\sigma_L(b) \setminus \sigma_L(a)) \neq \emptyset} C_i$. [(4.2), (4.9)]

(ii) $\bigcup_{C_i \cap (V \setminus U) \neq \emptyset} C_i \subseteq (\nabla^* \sigma_L(b) \setminus \nabla^* \sigma_L(a)) \cup \; (\triangle^* \sigma_L(b) \setminus \triangle^* \sigma_L(a))$:

(1) $x \in \bigcup_{C_i \cap (V \setminus U) \neq \emptyset} C_i$, with C_i maximal chain in $\mathcal{I}_p(L)$,

(2) there is i_0 such that $x \in C_{i_0}$ and $C_{i_0} \cap (V \setminus U) \neq \emptyset$, [(1)]

(3) $x \in V \setminus U$ or $x \notin V \setminus U$. [(2)]

If in (3)

(4) $x \in V \setminus U$,

two cases may occur:

(4.1) (a) $max\mathcal{I}_p(L)$ or (b) $x \in min\mathcal{I}_p(L)$, [$\mathcal{I}_p(L)$ is an M_3-space]

if in (4.1), occurs (a)

(4.2) $x \in \triangle^* V$, [(4.1)(a), (DP2) (D)]

(4.3) $x \notin \triangle^* U$, [(4), $\triangle^* U \subseteq U$]

(4.4) $x \in \triangle^* V \setminus \triangle^* U$, [(4.2), (4.3)]

if in (4.1), occurs (b)

(4.5) $x \in m_{\mathcal{I}_p(L)} V$, [(4), (4.1)(b)]

(4.6) $x \in \neg V \subseteq \nabla^* V$, [(4.5), (DP2) (B)]

if

(4.7) $x \in \nabla^* U$,

(4.8) $x \in \neg U$, [(4), (4.7)]

(4.9) $x \in m_{\mathcal{I}_p(L)\mathcal{I}_p(L)} U$, [(4.8), (4.5)]

(4.10) $x \in U$, [(4.9)]

(4.11) (4.10) contradicts (4),

(4.12) $x \notin \nabla^* U$, [(4.7), (4.11)]

(4.13) $x \in \nabla^* V \setminus \nabla^* U$, [(4.6), (4.12). $\mathcal{I}_p(L)$ is an M_3-space]

If in (3)

(5) $x \notin V \setminus U$,

(5.1) $x \in C_{i_0} = C_y$ with $y \in V \setminus U$, [(2), (5)]

working analogously to the case (4), from (5.1) we have:

(5.2) $y \in \triangle^* V \setminus \triangle^* U$ or $y \in \nabla^* V \setminus \nabla^* U$, [(5.1)]

taking into account that sets $\triangle^* Z$ and $\nabla^* Z$ are $\triangle-$involutive for every subset Z of the space, it turns out that

(5.3) $x \in \triangle^* V \setminus \triangle^* U$ or $x \in \nabla^* V \setminus \nabla^* U$. [(5.1), (5.2)]

■

Corollary 3.16 *Let $L \in \mathbf{M_3}$, $\mathcal{I}_p(L)$ be the M_3-space associated with L and $a, b \in L$ such that $a \leq b$. If $G \in O_\triangle(\mathcal{I}_p(L))$, then the following conditions are equivalent:*

(i) $\Theta_{O\triangle}(G) = \Theta(a, b)$,

(ii) $G = \sigma_L(d)$ *with* $d = (\nabla b \wedge \overline{\nabla a}) \vee (\triangle b \wedge \overline{\triangle a})) \in K(L)$, *where* $\overline{\nabla a}$ *and* $\overline{\triangle a}$ *are the Boolean complement in* $K(L)$ *of* ∇a *and* $\triangle a$ *respectively.*

Proof. It is an immediate consequence of Propositions 3.6 and 3.15. ∎

Remark 3.17 *As a consequence of* Proposition 3.6, *the set* $G = \sigma_L(d)$ *with* $d \in K(L)$, *is an open, closed and* $\triangle-$*involutive subset of* $\mathcal{I}_p(L)$.

Proposition 3.18 *Let $L \in M_3$, let $d \in K(L)$ and $\overline{d}$ be the Boolean complement of d in $K(L)$. Then the following conditions are equivalent for all $\mathbf{M_3}-$congruence φ on L:*

(i) $\varphi = \Theta_{O\triangle}(\sigma_L(d))$,

(ii) $\varphi = \Theta_{C\triangle}(\sigma_L(\overline{d}))$,

(iii) $\varphi = \theta(I(d))$, *where* $\theta(I(d))$ *is the congruence associated with the ideal* $I(d)$

Proof. (i) $\Leftrightarrow$ (ii): It results from Theorems 1.1 and 2.4, Lemma 2.3 and Observation 3.17.

(ii) $\Rightarrow$ (iii): Let $\varphi = \Theta_{C\triangle}(\sigma_L(\overline{d}))$, with $d \in K(L)$. Then $Y = \sigma_L(\overline{d})$ is a closed and increasing subset of $\mathcal{I}_p(L)$ and by Proposition 3.2, there is the ideal $I = \bigcap\limits_{Q \in Y} Q$ such that $Y = \sigma(I)$. Let us see what $I = I(d)$.

Indeed, let $x \in I(d)$ and $Q \in Y$, then $Q \in \mathcal{I}_p(L)$ and $\overline{d} \notin Q$. Since $0 = d \wedge \overline{d} \in Q$, for all $Q \in Y$, we have that $d \in Q$ for all $Q \in Y$ and so $x \in \bigcap\limits_{Q \in Y} Q$.

For the other inclusion, let $y \in \bigcap\limits_{Q \in Y} Q$, then it is clear that (1) $y, d \in Q$, for all $Q \in Y$. If $y \not\leq d$, then from Birkhoff-Stone theorem, there is an ideal prime S such that $d \in S$ and $y \notin S$. Then $S \in \sigma_L(\overline{d}) = Y$ and $y \notin S$, which contradicts (1). Hence, $y \in I(d)$.

Consequently, $Y = \sigma(I(d))$, whence $\varphi = \theta(I(d))$ by Proposition 3.1.

(iii)$\Rightarrow$ (ii): Let $d \in K(L)$ and $\varphi = \theta(I(d))$, being $\theta(I(d))$ the congruence associated with the ideal $I(d)$. Then by Proposition 3.1, we have that (1) $\varphi = \Theta(\sigma(I(d)))$.

Besides, (2) $\sigma(I(d)) = \sigma_L(\overline{d})$. Indeed: if $I \in \sigma(I(d))$, then $I \in \mathcal{I}_p(L)$ and $I(d) \subseteq I$. Then $d \in I$ and since I is an proper ideal, for being an prime ideal, it is fulfilled that $\overline{d} \notin I$. So $I \in \sigma_L(\overline{d})$. Reciprocally, if $I \in \sigma_L(\overline{d})$, then $I \in \mathcal{I}_p(L)$ and $\overline{d} \notin I$. Since $0 = d \wedge \overline{d}$, results $d \in I$. Consequently $I(d) \subseteq I$ and therefore $I \in \sigma(I(d))$.

Then, by (1), (2) and Observation 3.17, we infer that $\varphi = \Theta_{C\triangle}(\sigma_L(\overline{d}))$. ■

Proposition 3.19 *Let $L \in \mathbf{M_3}$ and $\mathcal{I}_p(L)$ be the M_3-space associated with L. If $G \in O_\triangle(\mathcal{I}_p(L))$, then the following conditions are equivalent:*

(i) $\Theta_{O\triangle}(G)$*, is a principal* $\mathbf{M_3}-$*congruence on* L*,*

(ii) G *is an open, closed and* $\triangle-$*involutive subset of* $\mathcal{I}_p(L)$*.*

Proof. (i) $\Rightarrow$ (ii): By Corollary 3.16, we know that if $\Theta_{O\triangle}(G)$ is a principal $\mathbf{M_3}-$congruence, say $\Theta_{O\triangle}(G) = \Theta(a,b)$, with $a \leq b$, then $G = \sigma_L(d)$ with $d = (\nabla b \wedge \overline{\nabla a}) \vee (\triangle b \wedge \overline{\triangle a}) \in K(L)$. From the latter, by Observation 3.17, we infer that G is an open, closed, and $\triangle-$involutive set of $\mathcal{I}_p(L)$.

(ii) $\Rightarrow$ (i): Let G be an open, closed and $\triangle-$involutive subset of $\mathcal{I}_p(L)$, then by Proposition 3.5, $G = V \setminus U$, where $U, V \in D(\mathcal{I}_p(L))$, $U \subseteq V$, $V = \sigma_L(b)$ and $U = \sigma_L(a)$, with $a, b \in L$. Since G is also $\triangle-$involutive subset, then by taking into account Lemma 2.1 it is verified, $G = \triangle^* V \setminus \triangle^* U$. On the other hand, by (DP2), we can prove that $\nabla^* V \setminus \nabla^* U \subseteq \triangle^* V \setminus \triangle^* U$. Then, $G = (\nabla^* \sigma_L(b) \setminus \nabla^* \sigma_L(a)) \cup (\triangle^* \sigma_L(b) \setminus \triangle^* \sigma_L(a))$, hence $\Theta_{O\triangle}(G)$ is a principal $\mathbf{M_3}-$congruence, by Proposition 3.15. ■

Theorem 3.20 *Let $L \in \mathbf{M_3}$ and let $\mathcal{I}_p(L)$ be the M_3-space associated with L. Then, the lattice $OC_\triangle(\mathcal{I}_p(L))$ of open, closed and $\triangle-$involutive subsets of $\mathcal{I}_p(L)$ is isomorphic to the lattice $Con_{\mathbf{M_3}P}(L)$ of principal $\mathbf{M_3}-$congruences on L, and the isomorphism is established by the function $\Theta_{OC\triangle} : OC_\triangle(\mathcal{I}_p(L)) \longrightarrow Con_{\mathbf{M_3}P}(L)$ defined by the same prescription as the function $\Theta_{O\triangle}$ given in* (A3')*.*

Proof. It is a consequence of Theorem 2.4, Proposition 3.19, and the fact that $OC_\triangle(\mathcal{I}_p(L))$ is a sublattice of $O_\triangle(\mathcal{I}_p(L))$. ■

Corollary 3.21 *The lattice of the principal congruences of a bounded M_3-lattice is a Boolean algebra.*

Proof. It is an immediate consequence of Theorem 3.20, taking into account that the lattice of open, closed and $\triangle-$involutive subsets of its associated M_3-space is a Boolean algebra. ■

Remark 3.22 *The following properties are satisfied in every bounded M_3-lattice:*

(i) *the intersection of a finite number of principal* $\mathbf{M_3}-$*congruences is also a principal* $\mathbf{M_3}-$*congruence,*

(ii) *the principal* $\mathbf{M_3}-$*congruences are Boolean.*

Proposition 3.23 *Let* $L \in \mathbf{M_3}$ *and let* $\mathcal{I}_p(L)$ *be the* M_3-*space associated with* L*. If* G *is a subset of* $\mathcal{I}p(L)$*, then the following conditions are equivalent:*

(i) G *is an open, closed and* $\triangle-$*involutive subset of* $\mathcal{I}p(L)$*,*

(ii) *there exists* $a \in K(L)$ *such that* $G = \sigma_L(a)$*.*

Proof. (i) $\Rightarrow$ (ii): Let G be an open, closed and $\triangle-$involutive subset of $\mathcal{I}p(L)$, then, in particular G is an open, closed and decreasing subset. Since σ_L is an isomorphism between L and $D(\mathcal{I}p(L))$, there is $a \in L$ such that $G = \sigma_L(a)$. On the other hand, as G is $\triangle-$involutive subset, it is verified that $G = \triangle^* G$, and consequently $G = \sigma_L(\triangle a)$ with $\triangle a \in K(L)$.

(ii) $\Rightarrow$ (i): By hypothesis, there is $a \in K(L)$ such that $G = \sigma_L(a)$. Then $a = \triangle a$ and $G = \sigma_L(\triangle a) = \triangle^* \sigma_L(a)$. Hence, G is an open, closed, and $\triangle-$involutive subset of $\mathcal{I}_p(L)$. ■

Theorem 3.24 *Let* $L \in \mathbf{M_3}$ *and let* $\mathcal{I}_p(L)$ *be the* M_3-*space associated with* L*. Then, the lattice* $K(L)$ *of Boolean elements of* L *is isomorphic to the lattice* $OC_\triangle(\mathcal{I}_p(L))$ *of open, closed and* $\triangle-$*involutive subsets of* $\mathcal{I}_p(L)$*, and the isomorphism is defined by the restriction to* $K(L)$ *of isomorphism* $\sigma_L : L \longrightarrow D(\mathcal{I}_p(L))$*, defined as in* (A1)*.*

Proof. Immediate from (DP1) of Section 1, and Proposition 3.23. ■

Corollary 3.25 *Let* $L \in \mathbf{M_3}$ *and let* $\mathcal{I}_p(L)$ *be the* M_3-*space associated with* L*. Then, the lattice* $K(L)$ *of Boolean elements of* L *is isomorphic to the lattice* $Con_{\mathbf{M_3}P}(L)$ *of principal* $\mathbf{M_3}-$*congruences on* L*, and the isomorphism is the composition* $\Theta_{OC\triangle} \circ \sigma_L$*.*

Proof. Immediate from Theorems 3.20 and 3.24. ■

The following corollary provides a characterization of the congruences on the finite M_3-lattices.

Corollary 3.26 *The* $\mathbf{M_3}-$*congruences on a finite* M_3-*lattice are principal.*

Proof. Let L be a finite M_3-lattice and let φ be an $\mathbf{M_3}-$congruence on L. Then by Theorem 2.4, there is an open and $\triangle-$involutive subset G of $\mathcal{I}p(L)$ such that $\varphi = \Theta_{O\triangle}(G)$. On the other hand as L is finite, then $\mathcal{I}_p(L)$ is the cardinal sum of a finite number of two-elements chains and the Priestley space topology is the discrete.

Then G is open, closed and $\triangle-$involutive set and consequently the congruence that determines, by Theorem 3.20, is a principal $\mathbf{M_3}-$congruence. ■

Corollary 3.27 *Let L be a bounded M_3-lattice such that its associated M_3-space is the cardinal sum of n chains, with n a positive integer. If $K(L)$ is the lattice of Boolean elements of L, then $|Con_{\mathbf{M_3}}(L)| = |K(L)| = 2^n$, where $|Z|$ denotes the cardinality of the Z set.*

Proof. Let L be a bounded M_3-lattice in the conditions of the theorem. Then L is a finite set and consequently, by Corollary 3.26, the congruences on L are principal.

On the other hand, from Theorem 3.20 and Proposition 3.13, each principal $\mathbf{M_3}-$congruence on L is determined by a subset of $\mathcal{I}_p(L)$, which is a finite union of two-element chains. Then taking into account Corollary 3.25, we conclude that $|K(L)| = |Con_{\mathbf{M_3}P}(L)| = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$. ■

Corollary 3.28 *Let L be a finite M_3-lattice with n Boolean elements (i.e. $|K(L)| = n$), then its M_3-space associated is a cardinal sum of $Log_2 n$ two-element chains.*

Proof. It follows immediately from Corollary 3.27. ■

Finally we were able to determine that the principal $\mathbf{M_3}-$congruences on an M_3-lattice are the congruences associated to the ideals generated by the Boolean elements of this algebra, as the following result shows:

Proposition 3.29 *Let $L \in \mathbf{M_3}$, $\mathcal{I}_p(L)$ be the M_3-space associated with L and $a, b \in L$ such that $a \leq b$. If $\overline{\nabla a}$ and $\overline{\triangle a}$ are the Boolean complements in $K(L)$ of ∇a and $\triangle a$, respectively, then the following conditions are equivalent:*

(i) $\Theta(a, b) = \Theta_{O\triangle}(G)$,

(ii) $\Theta(a, b) = \Theta_{O\triangle}(\sigma_L(d))$, *with* $d = (\nabla b \wedge \overline{\nabla a}) \vee (\triangle b \wedge \overline{\triangle a})) \in K(L)$,

(iii) $\Theta(a, b) = \theta(I(d))$, *with* $d = (\nabla b \wedge \overline{\nabla a}) \vee (\triangle b \wedge \overline{\triangle a})) \in K(L)$, *where $\theta(I(d))$ is the congruence associated to the ideal $I(d)$.*

Proof. It is a direct consequence from Corollary 3.16 and Proposition 3.18. ■

Corollary 3.30 *Let $L \in \mathbf{M_3}$ and let φ be a congruence on L. Then the following conditions are equivalent:*

(i) *φ is a principal $\mathbf{M_3}-$congruence on L,*

(ii) *$\varphi = \theta(I(d))$ with $d \in K(L)$, where $\theta(I(d))$ is the congruence associated with the ideal $I(d)$.*

Proof. It follows immediately by Theorem 2.4 and Proposition 3.29. ■

Corollary 3.31 *Every bounded M_3-lattice has the principal $\mathbf{M_3}-$congruences equationally definable* (CPDE)*.*

Proof. It is immediate from Corollary 3.30. ■

An important consequence of the above proposition is the following:

Proposition 3.32 *If $L \in \mathbf{M_3}$, φ_1 and φ_2 are principal $\mathbf{M_3}-$congruences on L such that $\varphi_1 = \theta(I(d))$ and $\varphi_2 = \theta(I(k))$ with $d, k \in K(L)$, then the following properties are verified:*

(i) *$\varphi_1 \vee \varphi_2 = \theta(I(d \vee k))$,*

(ii) *$\varphi_1 \circ \varphi_2 = \varphi_1 \vee \varphi_2$.*

Proof. Let φ_1 and φ_2 be principal $\mathbf{M_3}-$congruences on L such that (1) $\varphi_1 = \theta(I(d))$, (2) $\varphi_2 = \theta(I(k))$, with $d, k \in K(L)$. By Proposition 3.18, we have that $\varphi_1 \vee \varphi_2 = \Theta_{O\triangle}(\sigma_L(d)) \vee \Theta_{O\triangle}(\sigma_L(k))$ and as $\Theta_{O\triangle}$ and σ_L are isomorphisms, we obtain (3) $\varphi_1 \vee \varphi_2 = \theta(I(d \vee k))$.

On the other hand, we prove that $\varphi_1 \circ \varphi_2 = \varphi_1 \vee \varphi_2$. Indeed, let $(x, y) \in \varphi_1 \circ \varphi_2$, then there exists $z \in L$ such that $(x, z) \in \varphi_2$ and $(z, y) \in \varphi_1$. Then, taking into account (1) and (2), it is fulfilled that $x \vee k = z \vee k$ and $z \vee d = y \vee d$, whence $x \vee d \vee k = z \vee d \vee k$ and $z \vee d \vee k = y \vee d \vee k$. So $x \vee d \vee k = y \vee d \vee k$, from which we infer that $(x, y) \in \theta(I(d \vee k))$ and from (3) we then conclude that $(x, y) \in \varphi_1 \vee \varphi_2$.

The other inclusion is immediate and it results from the fact that $\varphi_i \subseteq \varphi_1 \circ \varphi_2$ for $i = 1, 2$ and therefore $\varphi_1 \vee \varphi_2 \subseteq \varphi_1 \circ \varphi_2$. ■

Corollary 3.33 *In the variety* $\mathbf{M_3}$*, the composition of principal congruences is commutative.*

Proof. It is an immediate consequence of Proposition 3.32. ■

4 Boolean M_3-congruences

The following results give a characterization of Boolean congruences on M_3-lattices.

Lemma 4.1 *Let* $L \in \mathbf{M_3}$*,* $\mathcal{I}_p(L)$ *be the* M_3-*space associated with* L *and* Y *be an open and* $\triangle-$*involutive subset of* $\mathcal{I}_p(L)$*. Then the following conditions are equivalent:*

(i) $\Theta_{O\triangle}(Y)$ *is a Boolean* $\mathbf{M_3}-$*congruence on* L*,*

(ii) $\mathcal{I}_p(L) \setminus Y$ *is an open and* $\triangle-$*involutive subset of* $\mathcal{I}_p(L)$*.*

Proof. (i) $\Rightarrow$ (ii): Let $\Theta_{O\triangle}(Y)$ be a Boolean $\mathbf{M_3}-$congruence on L. Then, there is $\Theta_{O\triangle}(G) \in Con_{\mathbf{M_3}}(L)$, such that $\Theta_{O\triangle}(Y) \cap \Theta_{O\triangle}(G) = id_L$ and $\Theta_{O\triangle}(Y) \cup \Theta_{O\triangle}(G) = L \times L$, being Y and G open and $\triangle-$involutive subsets of $\mathcal{I}_p(L)$.

Since $\Theta_{O\triangle}$ is an isomorphism it is verified that $\Theta_{O\triangle}(Y \cap G) = \Theta_{O\triangle}(\emptyset)$ and $\Theta_{O\triangle}(Y \cup G) = \Theta_{O\triangle}(\mathcal{I}_p(L))$. Hence, $Y \cap G = \emptyset$ and $Y \cup G = \mathcal{I}_p(L)$, and therefore $G = \mathcal{I}_p(L) \setminus Y$, which implies that $\mathcal{I}_p(L) \setminus Y$ is an open and $\triangle-$involutive set.

(ii) $\Rightarrow$ (i): Let $G = \mathcal{I}_p(L) \setminus Y$ be an open and $\triangle-$involutive subset of $\mathcal{I}_p(L)$, then $\Theta_{O\triangle}(G) \in Con_{\mathbf{M_3}}(L)$. From the Theorem 3.20, Y is also an open and $\triangle-$involutive set, therefore we have to $\Theta_{O\triangle}(Y) \in Con_{\mathbf{M_3}}(L)$. Besides, for being $\Theta_{O\triangle}$ an isomorphism, we have that $\Theta_{O\triangle}(Y) \cap \Theta_{O\triangle}(G) = \Theta_{O\triangle}(Y \cap G) = \Theta_{O\triangle}(\emptyset) = id_L$ and $\Theta_{O\triangle}(Y) \cup \Theta_{O\triangle}(G) = \Theta_{O\triangle}(Y \cap G) = \Theta_{O\triangle}(\mathcal{I}_p(L)) = L \times L$, which implies $\Theta_{O\triangle}(Y)$ is a Boolean $\mathbf{M_3}-$congruence. ■

Proposition 4.2 *Let* $L \in \mathbf{M_3}$ *and let* $\mathcal{I}_p(L)$ *be the* M_3-*space associated with* L*. If* Y *is a subset of* $\mathcal{I}_p(L)$*, then the following conditions are equivalent:*

(i) $\Theta_{O\triangle}(Y)$ *is a Boolean* $\mathbf{M_3}-$*congruence on* L*,*

(ii) Y *is an open, closed, and* $\triangle-$*involutive subset of* $\mathcal{I}_p(L)$*.*

Proof. (i) $\Rightarrow$ (ii): If $\Theta_{O\triangle}(Y)$ is a Boolean $\mathbf{M_3}-$congruence on L, then Y is an open and $\triangle-$involutive subset of $\mathcal{I}_p(L)$. By Lemma 4.1, it is verified that $\mathcal{I}p(L) \setminus Y$ is also an open and $\triangle-$involutive subset of $\mathcal{I}_p(L)$, then we have that Y is an open, closed and $\triangle-$involutive subset of $\mathcal{I}_p(L)$.

(ii) $\Rightarrow$ (i): Let Y be an open, closed, and $\triangle-$involutive subset of $\mathcal{I}_p(L)$. Then $\mathcal{I}_p(L)\backslash Y$ is an open and $\triangle-$involutive subset and by Lemma 4.1, we can conclude that $\Theta_{O\triangle}(Y)$ is a Boolean $\mathbf{M_3}-$congruence. ■

Corollary 4.3 *Let $L \in \mathbf{M_3}$. Then the following conditions are equivalent:*

(i) *φ is a Boolean $\mathbf{M_3}-$congruence on L,*

(ii) *φ is a principal $\mathbf{M_3}-$congruence on L.*

Proof. It follows from Theorem 3.20 and Proposition 4.2. ■

Theorem 4.4 *Let $L \in \mathbf{M_3}$ and let $\mathcal{I}_p(L)$ be the M_3-space associated with L. Then, the lattice $OC_{\triangle}\mathcal{I}_p(L)$ of open, closed and $\triangle-$involutive subsets of $\mathcal{I}_p(L)$ is isomorphic to the lattice $Con_{\mathbf{M_3}B}(L)$ of Boolean $\mathbf{M_3}-$congruences on L, and the isomorphism is established by function $\Theta_{OC\triangle} : OC_{\triangle}(\mathcal{I}_p(L)) \longrightarrow Con_{\mathbf{M_3}B}(L)$ defined by the same prescription as the function $\Theta_{O\triangle}$ given in* (A3').

Proof. It is immediate by Corollary 4.3 and Theorem 3.20. ■

References

[1] A. V. Figallo. Los M_3-Reticulados. *Rev. Colombiana de Matemática*, 21:95–106, 1987. Zbl 065806005, MR0968380.

[2] A. V. Figallo. M_3-Reticulados finitos. *Rev. de la Unión Matematica Argentina*, 36:21–26, 1990. MR1265691.

[3] A. V. Figallo. Una nota sobre M_3-Reticulados. *Rev. Colombiana de Matemática*, 2:145-151, 1990. Zbl 072906010, MR1106602.

[4] A. V. Figallo and M. A. Jiménez. A topological duality for M_3-lattices. *Annals of the University of Craiova, Mathematics and Computer Science Series*, 45(2):174–189, 2018. MR3897640.

[5] A. Monteiro. Sur la définition des algèbres de Lukasiewicz trivalentes. *Bull. Math. Soc. Sci. Math. Phys, R. P. Roum*, 7(55):3–12, 1963. MR0191809.

[6] H. A. Priestley. Representation of distributive lattices by means of ordered Stone spaces. *Bull. London Math. Soc.*, 2:186–190, 1970. doi:10.1112/blms/2.2.186, Zbl 0201.01802, MR0265242.

[7] H. A. Priestley. Ordered topological spaces and the representation of distributive lattices. *Proc. London Math. Soc.*, 24(3):507–530, 1972. doi:10.1112/plms/s3-24.3.507, Zbl 0323.06011, MR0300949.

[8] H. A. Priestley. Ordered sets and duality for distributive lattices. *Ann. Discrete Math.*, 23:39–60, 1984. MR0779844.

[9] H. A. Priestley. Stone lattices: a topological approach. *Fund. Math.*, 84:127–143, 1974. Zbl 0323.06012, MR0340136.

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