South American Journal of Logic
Vol. X, n. X, pp. 1–15, 20XX
ISSN: 2446-6719

Congruences on bounded Hilbert algebras with Moisil possibility operators

María Cristina Canals Frau, Aldo V. Figallo and Gustavo Pelaitay

Abstract

In this paper, we will introduce the variety of bounded Hilbert algebras with Moisil possibility operators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$, called MI_n^0-algebras. First, we give a characterization of MI_n^0-congruences in terms of a particular class of deductive systems, namely modal deductive systems. Furthermore, from the above results on MI_n^0-congruences, the principal ones are described. In addition, we proved that the variety of MI_n^0-algebras is semisimple.

Keywords: Hilbert algebras; bounded Hilbert algebras; Moisil possibility operators.

Introduction

In 1923, David Hilbert proposed to study implicative fragment of intuitionistic propositional calculus. This fragment is well-known as *positive implicative calculus* and its study was begun in 1935 by D. Hilbert and P. Bernays.

In 1950, L. Henkin ([13]) introduced *implicative models* as algebraic models of positive implicative calculus. Later, A. Monteiro renamed it as *Hilbert algebras* and his Ph. D. student A. Diego ([10, 11, 12]) made one of the most important contributions to this algebraic structure which we can define as follow:

A Hilbert algebra (or I-algebra) is an algebra $\langle A, \to, 1\rangle$ of type $(2, 0)$ such that the following axioms hold in A :

(I1) $1 \to x = x$,

(I2) $x \to x = 1$,

(I3) $x \to (y \to z) = (x \to y) \to (x \to z)$, (I4) $(x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y)$.

The variety of Hilbert algebras is denoted by $\mathcal{I}$. For each $A \in \mathcal{I}$ the following properties are verified:

(I5) $x \to 1 = 1$,

(I6) the binary relation $\leq$ defined by $x \leq y$ if and only if $x \to y = 1$ is a partial order on A with greatest element 1.

(I7) $x \to (y \to z) = y \to (x \to z)$,

(I8) $x \leq y$ implies $y \to z \leq x \to z$,

(I9) $x \to (y \to x) = 1$,

(I10) $x \leq y$ implies $z \to x \leq z \to y$.

A. Monteiro ([16]), proved that the semisimple I-algebras are those that verify the additional identity:

(I11) $(x \to y) \to x = x$.

This author called *Tarski algebras* to semisimple I-algebras and Pierce law to identity I11.

(I12) Let A be an I-algebra and let $t \in A$. We say that $t \in A$ is a tarskian element of A if t satisfies the identity:

(T) $(t \to x) \to t = t$ for all $x \in A$,

The set of all tarskian elements of an I-algebra A is denoted by $T(A)$.

Let A be a Hilbert algebra. A subset $D \subseteq A$ is a deductive system of A ([2, 14]) if $1 \in D$ and if $x, x \to y \in D$, then $y \in D$. The set of all deductive systems of a Hilbert algebra A is denoted by $\mathcal{D}(A)$.

Other interesting properties of I-algebras are the following:

(I13) The deductive system generated by a set $X \subseteq A$ is $[X) := \bigcap\{D \in \mathcal{D}(A) : X \subseteq D\}$. In particular, if $X = \{a\}$, the principal deductive system is $[a) = \{x \in A : a \leq x\}$.

(I14) If A is an I-algebra and $Con_{\mathbf{I}}(A)$ is the set of all **I**-congruences of A, then $Con_{\mathbf{I}}(A) = \{R(D) : D \in \mathcal{D}(A)\}$ where $R(D) = \{(x, y) \in A^2 : x \to y \in D,\ y \to x \in D\}$. Besides, $[1]_{R(D)} = D$ and if $\Theta \in Con_{\mathbf{I}}(A)$, then $R([1]_\Theta) = \Theta$.

A bounded Hilbert algebra (see [3, 5]) is a Hilbert algebra A with an element $0 \in A$ such that $0 \to a = 1$, for every $a \in A$. The notation a^* means $a \to 0$.

The following result has been proved by Buşneag in [2, 4].

(I15) Let A be a bounded Hilbert algebra. Then, the following conditions are equivalent:

(i) A is a Boolean lattice,

(ii) for all $x \in A$, $x^{**} = x$.

1 $\mathbf{MI_n}$-algebras

Gr. C. Moisil introduced the 3-valued Łukasiewicz algebras as algebraic models of 3-valued Łukasiewicz propositional calculus. It is well known that in 3-valued Łukasiewicz algebras it is possible to define an implication operator which shows that 3-valued Łukasiewicz algebras are a special case of Hilbert algebras. This result was, in some way, the motivation of the papers [6] and [7].

L. Iturrioz introduced in [15] the notion of modal operators on symmetric Heyting algebras and defined the class of SH_n-algebras. In [7, 8] Canals Frau and Figallo consider some reducts of this class. In particular, they introduced the following definition.

A Hilbert algebra of order n, $(n \geq 2)$, with the Moisil possibility operators (or MI_n-algebra) is an algebra $\langle A, \to, \sigma_1, \ldots, \sigma_{n-1}, 1\rangle$ of type (2,1,. . . ,1,0) such that the reduct $\langle A, \to, 1\rangle$ is a I-algebra and σ_1, . . ., σ_{n-1} are unary operations satisfying the following axioms:

(M1) $(\sigma_1 x \to y) \to x = x$,

(M2) $\sigma_i(x \to y) \to (\sigma_i x \to \sigma_j y) = 1$, $1 \leq i \leq j \leq n-1$,

(M3) $(\sigma_i x \to \sigma_i y) \to ((\sigma_{i+1} x \to \sigma_{i+1} y) \to \ldots ((\sigma_{n-1} x \to \sigma_{n-1} y) \to \sigma_i(x \to y))\ldots) = 1$, (M4) $\sigma_i(x \to \sigma_j y) = x \to \sigma_j y$, $1 \leq i, j \leq n-1$,

(M5) $\sigma_{n-1}x = (x \to \sigma_i x) \to \sigma_j x$, $1 \leq i \leq j \leq n-1$.

From now on, we will denote by $\mathcal{MI}_n$ the variety of MI_n-algebras.

Remark 1.1 *In [7] MI_n-algebras were called $(n+1)$-valued modal Hilbert algebras, following the terminology of Iturrioz we have called them Hilbert algebras of order n with Moisil operators.*

Now, we will summarize some useful properties of MI_n-algebras (see [7]).

(M6) $\sigma_1 x \leq x$,

(M7) $\sigma_i(\sigma_j x) = \sigma_j x$,

(M8) $\sigma_j 1 = 1$,

(M9) $\sigma_1 x \leq \sigma_2 x \leq \ldots \leq \sigma_{n-1}x$,

(M10) $x \leq \sigma_{n-1}x$,

(M11) $x \leq y$ implies $\sigma_i x \leq \sigma_i y$,

(M12) $\sigma_i(\sigma_j x \to y) = \sigma_j x \to \sigma_i y$, $i \leq j$,

(M13) $x \to \sigma_j(x \to y) = \sigma_j(x \to y)$,

(M14) $x \to \sigma_j y \leq \sigma_j(x \to y)$,

(M15) $\sigma_j(x \to y) \leq \sigma_j x \to \sigma_j y$,

(M16) $(\sigma_1 x \to \sigma_1 y) \to ((\sigma_2 x \to \sigma_2 y) \to \ldots((\sigma_{n-1}x \to \sigma_{n-1}y) \to (x \to y))\ldots) = 1$,

(M17) $\sigma_i x = \sigma_i y$ for all $i = 1, 2, \ldots, n-1$, implies $x = y$,

(M18) $(\sigma_j x \to y) \to \sigma_j x = \sigma_j x$,

(M19) $\sigma_{n-1}x = (x \to \sigma_1 x) \to x$,

(M20) $\sigma_1(\sigma_1 y \to x) \to (\sigma_1(\sigma_1 x \to z) \to (\sigma_1 y \to z)) = 1$.

(M21) The algebra $\mathbf{C_n}^{MI} = \langle \mathcal{C}_n, \to, \sigma_1, \ldots, \sigma_{n-1}, 1\rangle$, where $\mathcal{C}_n = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$,

$$x \to y = \begin{cases} 1 & \text{if } x \leq y, \\ y & x > y, \end{cases} \quad \text{and} \quad \sigma_j(\tfrac{k}{n-1}) = \begin{cases} 0 & \text{if } k + j \leq n-1, \\ 1 & \text{if } k + j > n-1 \end{cases} \quad 0 \leq k \leq n-1,$$

is a MI_n-algebra, called the standard MI_n-algebra.

In the MI_k-algebra $C_{\mathbf{k}}^{MI}$ with $2 \leq k < n-1$ we can define $\sigma_k, \sigma_{k+1}, \ldots, \sigma_{n-1}$ being $\sigma_k = \sigma_{k+1} = \ldots = \sigma_{n-1}$. Hence, the chain $C_{\mathbf{k}}^{MI} \in \mathcal{MI}_n$.

(M22) Let $A \in \mathcal{MI}_n$. $D \in \mathcal{D}(A)$ is a modal deductive system if it satisfies the following condition: $x \in D$ implies $\sigma_1 x \in D$.

The set of all modal deductive system of a MI_n-algebra A it is denoted by $\mathcal{D}_m(A)$.

Let $A \in \mathcal{MI}_n$, $X \subseteq A$ and $a \in A \setminus X$. $D_m(X)$ denotes the modal deductive system of A generated by X and $D_m(X, a)$ denotes the modal deductive system of A generated by $X \cup \{a\}$. Moreover, if B is a subalgebra of A we will denote $B \lhd A$.

Next, for the purpose of describing properties of modal deductive system we use the following notation introduced by Buşneag in ([2]) and frequently used by different authors:

$$(x_1, \ldots, x_{n-1}; x_n) = \begin{cases} x_n & \text{if} \quad n = 1 \\ x_1 \to (x_2, \ldots, x_{n-1}; x_n) & \text{if} \quad n > 1 \end{cases}.$$

(M23) Let $A \in \mathcal{MI}_n$, $X \subseteq A$ and $a \in A$. Then, the following conditions are verified:

(i) $D_m(X) = \{x \in A : \exists\, h_1, \ldots, h_k \in X : (\sigma_1 h_1, \ldots, \sigma_1 h_k; x) = 1\}$,

(ii) $D_m(a) = \{x \in A : (\sigma_1 a; x) = 1\} = [\sigma_1 a)$.

(iii) $D_m(X \cup \{a\}) = \{x \in A : (\sigma_1 a; x) \in D_m(X)\}$.

On the other hand, it is easy to see that:

(M24) If $A \in \mathcal{MI}_n$, $B \lhd A$ and $D_B \in \mathcal{D}_m(B)$. Then, there exists $D \in \mathcal{D}_m(A)$ such that $D_B = D \cap B$.

(M25) Let $A \in \mathcal{MI}_n$ and $M \in \mathcal{D}_m(A)$. Then, the following conditions are equivalents:

(i) M is a maximal,

(ii) A/M is a simple MI_n-algebra,

(iii) $A/M \simeq S \lhd \mathbf{C_n^{MI}}$.

2 Bounded $\mathbf{MI_n}$-algebras

In this section we are going to introduce the variety of bounded Hilbert algebras with Moisil possibility operators.

Definition 2.1 *A bounded MI_n-algebra (or MI_n^0-algebra) is an algebra $\langle A, \to, \sigma_1, \ldots, \sigma_{n-1}, 0, 1\rangle$ of type $(2, 1, \ldots, 1, 0, 0)$ where $\langle A, \to, \sigma_1, \ldots, \sigma_{n-1}, 1\rangle$ is a MI_n-algebra and it satisfies the following additional condition:*

(A1) $0 \to x = 1$.

We will denote by $\mathcal{MI}_n^0$ the variety of MI_n^0-algebras.

Example 2.2 *The algebra $C_{\mathbf{n}}^{MI^0} = \langle \mathcal{C}_n, \to, \sigma_1, \ldots, \sigma_{n-1}, 0, 1\rangle$ considered in* (M21) *is a MI_n^0-algebra.*

We will list some basic properties valid in the MI_n^0-algebras, proving just some of them.

Proposition 2.3 *Let $A \in \mathcal{MI}_n^0$. Then, the following properties are satisfied:*

(A2) $\sigma_i 0 = 0$,

(A3) $\sigma_i x^* = x^*$,

(A4) $\sigma_j(\sigma_i x)^* = (\sigma_i x)^*$,

(A5) $(\sigma_i x)^* \to \sigma_i x = \sigma_i x$,

(A6) $\sigma_i((\sigma_i x)^* \to \sigma_i y) = (\sigma_i x)^* \to \sigma_i y$,

(A7) $(\sigma_i x)^{**} = \sigma_i x$,

(A8) $(\sigma_i x)^* \to (\sigma_i y)^* = \sigma_i y \to \sigma_i x$,

(A9) $x^* = (\sigma_{n-1} x)^*$,

(A10) $\sigma_i x^* = (\sigma_{n-1} x)^*$,

(A11) $\sigma_i(\sigma_1 x)^* = (\sigma_1 x)^*$,

(A12) $(\sigma_1 x \to (\sigma_1 y)^*)^* \to x = 1$.

Proof.

(A2): From A1 and M6, we have that $\sigma_1 0 = 0$. Then, from M19, we infer that $\sigma_{n-1} 0 = (0 \to \sigma_1 0) \to 0 = 0$. Hence, from M9, we conclude that $\sigma_i 0 = 0$.

(A6): From A4, we have that $\sigma_i((\sigma_i x)^* \to \sigma_i y) = \sigma_i(\sigma_i((\sigma_i x)^*) \to \sigma_i y)$. Hence, taking into account M12 and M7, we obtain that $\sigma_i((\sigma_i x)^* \to \sigma_i y) = (\sigma_i x)^* \to \sigma_i \sigma_i y = (\sigma_i x)^* \to \sigma_i y$.

(A7): From A1, I10 and M18, we have that $(\sigma_i x \to 0) \to 0 \leq (\sigma_i x \to 0) \to \sigma_i x = \sigma_i x$. So, $(\sigma_i x)^{**} \leq \sigma_i x$. On the other hand, from I7 and I2 we obtain that $\sigma_i x \to (\sigma_i x)^{**} = \sigma_i x \to ((\sigma_i x \to 0) \to 0) = (\sigma_i x \to 0) \to (\sigma_i x \to 0) = 1$ from which we get that $\sigma_i x \leq (\sigma_i x)^{**}$.

(A9): From M10 and I8, we have that $(\sigma_n x)^* \leq x^*$. On the other hand, since $\sigma_1 x \leq x$ and so, from I8, we obtain that $x^* \to (\sigma_1 x)^* = (x \to 0) \to (\sigma_1 x \to 0) = 1$. Besides, from I7 and M5 we have that $x^* \to (\sigma_{n-1} x)^* = (x \to 0) \to (\sigma_{n-1} x \to 0) = \sigma_{n-1} x \to ((x \to 0) \to 0) = ((x \to \sigma_1 x) \to \sigma_1 x) \to ((x \to 0) \to 0)$. Hence, from I7, I3, A1 and I5, we obtain that $x^* \to (\sigma_{n-1} x)^* = (((x \to 0) \to (x \to \sigma_1 x)) \to ((x \to 0) \to \sigma_1 x)) \to ((x \to 0) \to 0) = ((x \to (0 \to \sigma_1 x)) \to ((x \to 0) \to \sigma_1 x)) \to ((x \to 0) \to 0) = ((x \to 0) \to \sigma_1 x) \to ((x \to 0) \to 0)$. For this result I3, M6 and I8 we obtain $x^* \to (\sigma_{n-1} x)^* = (x \to 0) \to (\sigma_1 x \to 0) = 1$. Therefore, $x^* \leq (\sigma_{n-1} x)^*$.

Definition 2.4 *An element x of a MI_n^0-algebra A is invariant if $\sigma_i x = x$.*

The set of all invariant elements of a MI_n^0-algebra A is denoted by $K(A)$.

Definition 2.5 *An element x of a MI_n^0-algebra A is regular if $x^{**} = x$.*

In what follows, the set of all regular elements of A we will denote by A^{**}.

Next, we will show the relationship between the above two definitions.

Proposition 2.6 *Let $A \in \mathcal{MI}_n^0$. Then, $K(A)$ is a MI_n^0-subalgebra of A.*

Proof. Let $x, y \in K(A)$. Then, $x = \sigma_i x$ and $y = \sigma_j y$, $1 \leq i, j \leq n-1$. Hence, from M12 we have that $x \to y = \sigma_i x \to \sigma_j y = \sigma_j(\sigma_i x \to y)$ and from M7 we deduce that $\sigma_k(x \to y) = x \to y$. Therefore, $x \to y \in K(A)$. On the other hand, from M7 $\sigma_k x = \sigma_k(\sigma_i x) = x$. So, $\sigma_k x \in K(A)$. Besides, from A2 and M8, we have that $0, 1 \in K(A)$.

Proposition 2.7 *Let $A \in \mathcal{MI}_n^0$. Then, $A^{**} = K(A)$.*

Proof. Let $x \in A^{**}$. Then, from A2 and M14, we have that $x = x^{**} = (x \to 0) \to \sigma_1 0 \leq \sigma_1 x^{**} = \sigma_1 x$. The other inequality results immediately from M6. Conversely, if $x \in K(A)$ then $x = \sigma_i x$. Then, from A7 we obtain that $x^{**} = (\sigma_i x)^{**} = \sigma_i x = x$. Therefore, $x \in A^{**}$.

Proposition 2.8 *Let $A \in \mathcal{MI}_n^0$. Then, $K(A)$ is a Boolean lattice.*

Proof. From the Proposition 2.6, we have that $\langle K(A), \to, \sigma_1, \ldots, \sigma_n, 0, 1\rangle$ is an MI_n^0-algebra. Hence, from A7 and I15 we obtain that $K(A)$ is a Boolean lattice.

Remark 2.9 *From Buşneag's proof of I15, it was proved that for every k_1, $k_2 \in K(A)$, the following properties hold:*

(i) $k_1 \vee k_2 = k_1^* \to k_2$,

(ii) k_1^* *is the boolean complement of* k_1.

Now, we will give another characterization of $K(A)$, using the tarskian elements of A.

Lemma 2.10 $T(A) = K(A)$.

Proof. Let $t \in T(A)$. Then, from M19, we have that $\sigma_{n-1}t = (t \to \sigma_1 t) \to t = t$. So, $t \in K(A)$. Conversely, let $k \in K(A)$ and $x \in A$. Then, we have that $(k \to x) \to k = (\sigma_i k \to x) \to \sigma_i k$ and from M18 we infer that $(k \to x) \to k = \sigma_i k = k$. Hence, $k \in T(A)$ and so, $K(A) = T(A)$.

3 Congruences

In this section we will determine the MI_n-congruences and we will establish a lattice isomorphism between $Con_{MI_n}(A)$ and $\mathcal{D}_m(A)$. Besides, we will obtain a characterization of MI_n-congruences. Furthermore, from the above results on the MI_n-congruences, the principal ones are described.

The following two theorems were stated in [7].

Theorem 3.1 *Let $A \in \mathcal{MI}_n$ and $D \in \mathcal{D}_m(A)$. Then, $Con_{MI_n}(A) = \{R(D) : D \in \mathcal{D}_m(A)\}$, where $R(D) = \{(x, y) \in A^2 : x \to y \in D,\ y \to x \in D\}$.*

Proof. Since A is a Hilbert algebra and D is a deductive system of A, by I14 we know that $R(D)$ is an I-congruence on A. Moreover, if $(x, y) \in R(D)$ since D is a modal deductive system, we have that $\sigma_1(x \to y), \sigma_1(y \to x) \in D$. Hence, from M9 we have that, $\sigma_i(x \to y)$, $\sigma_i(y \to x) \in D$, $1 \le i \le n-1$, and by M2 we infer that $\sigma_i x \to \sigma_i y$, $\sigma_i y \to \sigma_i x \in D$, $1 \le i \le n-1$. Therefore, $(\sigma_i x, \sigma_i y) \in D$, $1 \le i \le n-1$ from which we conclude that $R(D) \in Con_{MI_n}(A)$. Conversely, let $\theta \in Con_{MI_n}(A)$. Then, $\theta \in Con_I(A)$. From I14, we have that $[1]_\theta$ is a deductive system of A and $R([1]_\theta) = \theta$. Besides, from hypothesis and M8 we have that: if $(x, 1) \in \theta$, then $(\sigma_1 x, 1) \in \theta$, that is, $[1]_\theta \in \mathcal{D}_m(A)$ which completes the proof.

Theorem 3.2 *Let $A \in \mathcal{MI}_n$. Then, the lattices $Con_{MI_n}(A)$ and $\mathcal{D}_m(A)$ are isomorphic.*

Proof. It is a direct consequence of I14 and Theorem 3.1 considering the applications $\theta \mapsto [1]_\theta$ and $D \mapsto R(D)$ which are inverse to one another.

Next, we will show a characterization of simples MI_n^0-algebras.

Corollary 3.3 *Let $A \in \mathcal{MI}_n^0$. Then, the following conditions are equivalent:*

(i) *A is a simple MI_n^0-algebra,*

(ii) $\sigma_1(A) = \{0, 1\}$.

Proof. (i) $\Rightarrow$ (ii): Suppose that A is a simple MI_n^0-algebra and let $x \in A$. From (M23), we have that $[\sigma_1 x)$ is a modal deductive system of A. Hence, $[\sigma_1 x) = \{1\}$ or $[\sigma_1 x) = A$ from which it follows that $\sigma_1 x = 1$ or $\sigma_1 x = 0$.

(ii) $\Rightarrow$ (i): Suppose that $\sigma_1(A) = \{0, 1\}$. Let $D \in \mathcal{D}_m(A)$ and $x \in D$. Then, $\sigma_1 x \in D$. If $\sigma_1 x = 0$, we have that $D = A$ and if $\sigma_1 x = 1$, from M6, we have that $x = 1$. Therefore, $D = \{1\}$.

Let A be an MI_n-algebra and $a, b \in A$. By $\theta(a, b)$ we denote the principal congruence of A generated by (a, b), i.e., the smallest congruence of A that contains (a, b). In Theorem 3.4, we provide a description of the principal congruences of A.

Theorem 3.4 *Let $A \in \mathcal{MI}_n$. Then, for every $a, b \in A$ it is verified that: $\theta(a, b) = \{(x, y) \in A^2 : \sigma_1(a \to b) \to (\sigma_1(b \to a) \to x) = \sigma_1(b \to a) \to (\sigma_1(a \to b) \to y)\}$.*

Proof. Let $S = \{(x, y) \in A^2 : \sigma_1(a \to b) \to ((\sigma_1(b \to a) \to x) = \sigma_1(b \to a) \to ((\sigma_1(a \to b) \to y)\}$. Then, $(a, b) \in S$. Indeed, from M6, I5 and I3, we have that $1 = \sigma_1(b \to a) \to ((\sigma_1(a \to b) \to (a \to b)) = \big(\sigma_1(b \to a) \to ((\sigma_1(a \to b) \to a)\big) \to \big(\sigma_1(b \to a) \to ((\sigma_1(a \to b) \to b)\big)$. From this statement and I6 we have that $\sigma_1(b \to a) \to (\sigma_1(a \to b) \to a) \leq \sigma_1(b \to a) \to (\sigma_1(a \to b) \to b)$. In a similar way we obtain that $\sigma_1(a \to b) \to (\sigma_1(b \to a) \to b) \leq \sigma_1(a \to b) \to (\sigma_1(b \to a) \to a)$. Moreover, S is an equivalence relation on A such that:
(i): S is compatible with $\to$: Let $(x, y) \in S$ and $t \in A$. Then, we have that $\sigma_1(a \to b) \to (\sigma_1(b \to a) \to x) = \sigma_1(a \to b) \to (\sigma_1(b \to a) \to y)$. From this last statement, we obtain that $t \to (\sigma_1(a \to b) \to (\sigma_1(b \to a) \to x) = t \to (\sigma_1(a \to b) \to (\sigma_1(b \to a) \to y)$ and from I7 we obtain that $\sigma_1(a \to b) \to (\sigma_1(b \to a) \to (t \to x)) = \sigma_1(a \to b) \to (\sigma_1(b \to a) \to (t \to y))$. So, $(t \to x, t \to y) \in S$. Moreover, from I3, we have that $\sigma_1(a \to b) \to$

$(\sigma_1(b \to a) \to (x \to t)) = (\sigma_1(a \to b) \to (\sigma_1(b \to a) \to x)) \to (\sigma_1(a \to b) \to (\sigma_1(b \to a) \to t))$. From this last statement, I7 and I3 we deduce that $\sigma_1(a \to b) \to (\sigma_1(b \to a) \to (x \to t)) = (\sigma_1(b \to a) \to (\sigma_1(a \to b) \to y)) \to (\sigma_1(b \to a) \to (\sigma_1(a \to b) \to t)) = \sigma_1(b \to a) \to (\sigma_1(a \to b) \to (y \to t))$. Therefore, we conclude that $(x \to t, y \to t) \in S$.
(ii): S is compatible with σ_i : let $(x, y) \in S$. Then, $\sigma_1(a \to b) \to (\sigma_1(b \to a) \to x) = \sigma_1(b \to a) \to ((\sigma_1(a \to b) \to y)$ from which $\sigma_i\big(\sigma_1(a \to b) \to (\sigma_1(b \to a) \to x)\big) = \sigma_i\big(\sigma_1(b \to a) \to ((\sigma_1(a \to b) \to y)\big)$ and we conclude the proof by M12.

Hence, $S \in Con_{MI_n}(A)$. Finally, if $R \in Con_{MI_n}(A)$ and $(a, b) \in R$, then $S \subseteq R$. Indeed, let (1) $(x, y) \in S$. Since $(a, b) \in R$ we have that (2) $(\sigma_1(a \to b) \to x, x) \in R$ and $(\sigma_1(b \to a) \to y, y) \in R$ from which we obtain that $(\sigma_1(b \to a) \to (\sigma_1(a \to b) \to x), \sigma_1(b \to a) \to x) \in R$ and $(\sigma_1(a \to b) \to (\sigma_1(b \to a) \to y), \sigma_1(a \to b) \to y) \in R$. From (1) and I7, we conclude that $(\sigma_1(b \to a) \to x, \sigma_1(b \to a) \to y) \in R$. Hence, from (2), $(x, y) \in R$.

Corollary 3.5 *$\mathcal{MI}_n$ has equationally definable principal congruences.*

We prove that the variety $\mathcal{MI}_n$ satisfies the congruence extension property.

Lemma 3.6 *Let $A \in \mathcal{MI}_n$, $B \lhd A$ and $\theta \in Con_{MI_n}(B)$. Then, there exists $\varphi \in Con_{MI_n}(A)$ such that $\theta = \varphi \cap B^2$.*

Proof. Let $B \lhd A$ and $\theta \in Con_{MI}(B)$. Then, by Theorem 3.2, there exists $D_1 \in \mathcal{D}_m(B)$ such that $R(D_1) = \theta$. Hence, by (M24), there exists $D \in \mathcal{D}_m(A)$ such that $D \cap B = D_1$. Moreover, since $D \in \mathcal{D}_m(A)$ there exists $R(D) \in Con_{MI_n}(A)$. Let $\varphi = R(D)$ and suppose that $(x, y) \in \varphi \cap B^2$. Then, we have $x \to y$, $y \to x \in D \cap B$. So, $(x, y) \in R(D_1)$. From this last statement we have $\varphi \cap B^2 \subseteq \theta$. In a similar way we obtain that $\theta \subseteq \varphi \cap B^2$.

From Lemma 3.6 and a result of A. Day ([9]) the following property holds:

Lemma 3.7 *Let $A \in \mathcal{MI}_n$. Then, the following conditions are equivalent:*

(i) *$\mathcal{MI}_n$ satisfies the congruence extension property,*

(ii) *$\mathcal{MI}_n$ satisfies the principal congruences extension property,*

(iii) *for all A, $B \in \mathcal{MI}_n$ such that $B \lhd A$ and for all a, $b \in B$ it follows that $\theta_B(a, b) = \theta_A(a, b) \cap B^2$.*

Lemma 3.8 *$\mathcal{MI}_n$ has regular congruences.*

Proof. Let θ, $\varphi \in Con_{MI_n}(A)$ and $a \in A$ such that $[a]_\theta = [a]_\varphi$. Let us consider the quotients algebras A/θ and A/φ. Then, we have that: $[1]_\theta = [a]_\theta \to [a]_\theta = [a]_\varphi \to [a]_\varphi = [1]_\varphi$. Moreover, by Theorem 3.2, we have that $R([1]_\theta) = \theta$ and $R([1]_\varphi) = \varphi$. So, $\theta = \varphi$.

Lemma 3.9 *$\mathcal{MI}_n$ has distributive congruences.*

Proof. It is a direct consequence of [1] and taking into account that this variety has the EDPC property.

Let $A \in \mathcal{MI}_n^0$. We denote by $\mathcal{D}_m^P(A)$ the lattice of all principal deductive systems of a MI_n^0-algebra A and by $Con^P_{MI^0}(A)$ the lattice of all principal congruences of a MI_n^0-algebra A.

Lemma 3.10 *Let $A \in \mathcal{MI}_n^0$ and let a, $b \in A$. Then, $\big[w_{a,b}\big) \in \mathcal{D}_m^P(A)$, where $w_{a,b} := (\sigma_1(a \to b) \to (\sigma_1(b \to a))^*)^*$.*

Proof. From A4, we have that $\sigma_j w_{a,b} = w_{a,b}$, from which we conclude that $w_{a,b} \in K(A)$. From this last statement and from (ii) of (M23) we have that $[\,w_{a,b}) \in \mathcal{D}_m^P(A)$.

Proposition 3.11 *Let $A \in \mathcal{MI}_n^0$. Then, the lattices $K(A)$ and $\mathcal{D}_m^P(A)$ are anti-isomorphic.*

Proof. It follows from considering the application $\alpha : K(A) \longrightarrow \mathcal{D}_m^P(A)$ define by $\alpha(k) = [\,k)$ for all $k \in K(A)$.

In the following theorem we will obtain a good characterization of principal congruences in MI_n^0-algebras.

Theorem 3.12 *Let $A \in \mathcal{MI}_n^0$ and let a, $b \in A$. Then, $\theta(a,b) = \theta\big(w_{a,b}, 1\big)$.*

Proof. It is sufficient to show that:

(i) $\big(w_{a,b}, 1\big) \in \theta(a,b)$,

(ii) $(a,b) \in \theta\big(w_{a,b}, 1\big)$.

(i): From $(a,b) \in \theta(a,b)$ we infer that $(\sigma_1(b \to a), 1) \in \theta(a,b)$. Hence, we have that $(\sigma_1(a \to b) \to (\sigma_1(b \to a))^*, \sigma_1(a \to b) \to 0) \in \theta(a,b)$. From this last statement we have that $((\sigma_1(a \to b) \to (\sigma_1(b \to a))^*)^*, (\sigma_1(a \to b))^{**}) \in \theta(a,b)$ and by A7 we conclude that $((\sigma_1(a \to b) \to (\sigma_1(b \to a))^*)^*, \sigma_1(a \to b)) \in \theta(a,b)$. Besides, since $(\sigma_1(a \to b), 1) \in \theta(a,b)$ we have that $(w_{a,b}, 1) \in \theta(a,b)$.
(ii): (1) $(a \to b, b \to a) \in \theta(w_{a,b}, 1)$. Indeed, by Theorem 3.4, we must prove that $\sigma_1(w_{a,b} \to 1) \to (\sigma_1(1 \to w_{a,b}) \to (a \to b)) = \sigma_1(1 \to w_{a,b}) \to$

$(\sigma_1(w_{a,b} \to 1) \to (b \to a))$, which is equivalent to prove that $w_{a,b} \to (a \to b) = w_{a,b} \to (b \to a)$ and from A12 is equivalent to $1 = w_{a,b} \to (b \to a)$, which follows immediately from I7 and A12. Hence, from (1), I3, I2 and I1 we have that $\big((a \to b) \to ((b \to a) \to a), (b \to a) \to a\big) \in \theta(w_{a,b}, 1)$ and $\big((b \to a) \to ((a \to b) \to b), (a \to b) \to b\big) \in \theta(w_{a,b}, 1)$. From this last statement and I4 we deduce (2) $\big((a \to b) \to b, (b \to a) \to a\big) \in \theta(w_{a,b}, 1)$. On the other hand, by (1), we have that $(a \to b, 1) \in \theta(w_{a,b}, 1)$ and $(b \to a, 1) \in \theta(w_{a,b}, 1)$. So, we obtain $\big((a \to b) \to b, b\big) \in \theta(w_{a,b}, 1)$ and $\big((b \to a) \to a, b\big) \in \theta(w_{a,b}, 1)$. From these two above statements and (2) we conclude that $(a, b) \in \theta(w_{a,b}, 1)$.

Proposition 3.13 *Let $A \in \mathcal{MI}_n^0$. Then, the lattices $Con^P_{MI_n^0}(A)$ and $\mathcal{D}^P_m(A)$ are isomorphic.*

Proof. It follows from Theorem 3.12 and by considering the application $\Psi : Con^P_{MI_n^0}(A) \longrightarrow \mathcal{D}^P_m(A)$ defined by the prescription $\Psi\big(\theta(a, b)\big) = \big[w_{a,b}\big)$ for all $\theta(a, b) \in Con^P_{MI_n^0}(A)$.

Corollary 3.14 *Let $A \in \mathcal{MI}_n^0$. Then, the lattices $K(A)$ and $Con^P_{MI_n^0}(A)$ are anti-isomorphic.*

Proof. It is a direct consequence of Proposition 3.11 and 3.13.

Corollary 3.15 *Let A be a finite MI_n^0-algebra. Then, $|Con^P_{MI^0}(A)| = 2^m$ where m is the number of atoms of $K(A)$.*

Proof. It is a direct consequence of Proposition 3.14 and Proposition 2.8.

Next, we prove that the variety $\mathcal{MI}_n^0$ is semisimple.

Proposition 3.16 *Let $A \in \mathcal{MI}_n^0$. Then, the following conditions are equivalent:*

(i) *A is a subdirectly irreducible MI_n^0-algebra,*

(ii) *$K(A)$ has a unique dual atom,*

(iii) *A is a simple MI_n^0-algebra.*

Proof. (i) $\Rightarrow$ (ii): Let Θ_0 be a unique nontrivial minimal congruence. Then, there exists $(a, b) \in \Theta_0$ where $a \neq b$. Hence, $\theta(a, b) \subseteq \Theta_0$ and $\theta(a, b) \neq \Delta$ from which we get that $\Theta_0 = \theta(a, b)$. So, by Theorem 3.12 and Lemma 3.13, there exists a unique minimal principal modal deductive system $D_0 = [w_{a,b})$ where

$w_{a,b} \in K(A)$. Besides, taking into account that the lattices $K(A)$ and $\mathcal{D}_m^P(A)$ are anti-isomorphic we obtain that $w_{a,b}$ is a dual atom of $K(A)$.

(ii) $\Rightarrow$ (iii): Since A has a unique dual atom k, from Remark 2.9 we have that k^* is the unique atom of $K(A)$. Therefore, $K(A) = \{0, 1\}$ and from Theorem 3.3 we conclude that A is simple.

(iii) $\Rightarrow$ (i): It is easy to check.

Corollary 3.17 *$\mathcal{MI}_n^0$ is semisimple.*

Proof. It is a direct consequence of Proposition 3.16.

□

References

[1] R. Balbes and P. Dwinger, *Distributive Lattices*, University of Missouri Press, Columbia, 1974.

[2] D. Buşneag, A note on deductive systems of a Hilbert algebra, *Kobe J. Math.*, 2:29–35, 1985.

[3] D. Buşneag, On the maximal deductive systems of a bounded Hilbert algebra, *Bull. Math. de la Soc. Sci. Math. de la Roumanie*, 31:9–21, 1987.

[4] D. Buşneag, *Categories of algebraic logic*, Ed. Academiei Romane, 2006.

[5] D. Buşneag, M. Ghiţă. Some latticial properties of Hilbert algebras. *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)*, 53(101), no. 2:87–107, 2010.

[6] M. Canals Frau, A.V. Figallo, Modal three-valued Hilbert algebras, Instituto de Ciencias Básicas. Universidad Nacional de San Juan, Argentina, 1990.

[7] M. Canals Frau, A.V. Figallo, $(n+1)$-valued Hilbert modal algebras, *Notas de la Sociedad Matemática de Chile*, vol. X, Nro 1:143–149, Santiago de Chile, 1991.

[8] M. Canals Frau and A. V. Figallo, $(n+1)$-valued modal implicative semilattices, *1992 Proceedings The 22nd International Symposium on Multiple-Valued Logic*, 190–196. IEEE Computer Society, 1992.

[9] A. Day, A note on the congruence extension property, *Algebra Universalis*, 1:234–235, 1991.

[10] A. Diego, On implicative algebras, *Rev. Un. Mat. Argentina*, 20:310–311, 1962.

[11] A. Diego, Sobre las Álgebras de Hilbert, *Notas de Lógica Matemática*, 12, Universidad Nacional del Sur, Bahía Blanca, Argentina, 1965.

[12] A. Diego, Sur les algèbres de Hilbert, *Collection de Logique Mathématique*, Sér. A, Hermann, 21, 1966.

[13] L. Henkin, Fragments of the propositional calculus, *The Journal of Symbolic Logic*, 14(1):42–48, 1949.

[14] S. Hong and Y. Jun, On deductive systems of Hilbert Algebras, *Comm. Korean Math. Soc*, 11(3):595–600, 1996.

[15] L. Iturrioz. Modal operators on symmetrical Heyting algebras. *Universal algebra and applications* (Warsaw, 1978), 289–303. Banach Center Publ., 9, PWN, Warsaw, 1982.

[16] A. Monteiro, Algèbre de la Logique II, lecture notes from a course at Universidad Nacional del Sur, Bahía Blanca, Argentina, 1960.

María Cristina Canals Frau
Instituto de Ciencias Básicas
Universidad Nacional de San Juan
5400, San Juan, Argentina
E-mail: mcanalsfrau@gmail.com

Aldo V. Figallo
Instituto de Ciencias Básicas
Universidad Nacional de San Juan
5400, San Juan, Argentina
E-mail: avfigallo@gmail.com

Gustavo Pelaitay
Instituto de Ciencias Básicas
Universidad Nacional de San Juan
5400, San Juan, Argentina
E-mail: gpelaitay@gmail.com