South American Journal of Logic
Vol. 5, n. 1, pp. 1–21, 2019
ISSN: 2446-6719

First-Order Extensions for Public Announcement Logic

Marcio Kléos Freire Pereira

Abstract

Public Announcement Logics (*PAL*) have been widely known as the simplest versions of Dynamic Epistemic Logics (*DEL*). They were designed to model the reasoning about epistemic changes in groups of agents when these changes are triggered by a public simultaneous disclosure of some true information. Since their first formulations, almost the entirety of research on *PAL* has been oriented to their propositional level, and several open problems have been captivating the attention of many researchers from computer science and philosophy. In this paper, rather than dealing with open problems, we provide some complete axiomatizations for quantified *PAL*, along with related discussions.

Keywords: Dynamic Epistemic Logic, Public Announcement Logic, Quantified Epistemic Logic.

1 Introduction and motivations

Public Announcement Logics (*PAL*) are an umbrella description for a class of different systems formalizing changes in epistemic states of individual agents (or groups of agents), as long as these changes are triggered by a simultaneous and universal acquisition of a true information (an *announcement*) by all agents, provided that there is common knowledge that this acquisition is simultaneous and universal.

They were, so to speak, the first steps toward Dynamic Epistemic Logics (*DEL*). To be precise, we could view *PAL* as (usual) epistemic logics strenghtened with a single class of dynamic operators, not the other way round (dynamic logics containing epistemic modalities).[1]

[1]Roughly speaking, public announcements correspond to atomic programs or *tests* in dynamic logic, safeguarding their differences. We should remember that, as in epistemic logic with many agents we could count as many epistemic operators as many available agents,

This distinction is important, because in general, even when it comes to full-fledged *DEL* (which formalize other epistemic actions than public announcements, like private announcements, suspicions or individual acquisition of private informations, etc.), we are limited to reasoning about changes in the agents' *epistemic states*, without considering any *factual changes* in the world (as in typical dynamic logics).[2]

As it is well known, epistemic logic has to do with the reasoning about knowledge attribution to agents or groups of agents. Nevertheless, considering agents' epistemic states as static and immutable classes of propositions is a poor strategy for a logic of knowledge. Exchanging informations is an intrinsic part of knowledge (when understood as a process), and this is a sensible topic when it comes to epistemological or computational applications.

Inspired by the resources from dynamic logic, the *dynamic turn* in epistemic logics allowed the representation of epistemic changes in the object language itself, having *PAL* been the first attempts in that direction. These changes are mirrored in a semantics that, once a typical (relational) epistemic model is available, allowed straightforward and effective update strategies in order to calculate the semantic status of a proposition describing some epistemic actions and their outcomes.[3]

The two seminal works on *PAL* were [15] and [7], and each of them left its own mark on current standard treatments, and also has some feature which fell into disuse. The reader might consider reading [4, ch.4] for a nicely done presentation. A few extensions and alternative formulations have been proposed [8, 17, 18, 14], as well as contributions to open problems [10, 11].

In this paper, we are concerned with *first-order extensions for PAL* (or *FOPAL*). The standard approach for *PAL* consists, roughly speaking, in a sort of combination of Kripke-style relational semantics with update semantics. In a quantified setting, however, it is not trivial how to reasonably extend this strategy, for we must consider a number of features, including quantification domains, contingent identities and free individual variables.

As far as we know, only two sources have dealt with this topic: [13] and [12]. The former has provided only brief remarks on the subject, and the latter has chosen a rather complicated and non standard semantic framework, justified by author's specific concerns (basically: modelling his notion of *verifiable knowledge*). We also learn from Kishida [12] that Ma's preliminary results for *FOPAL* in [13] had an important mistake[4] and that no (complete) axioma-

strictly speaking, we actually have an infinite number of public announcement modalities, one for each proposition that could be announced.

[2]For a thorough presentation of dynamic logic, see [9]. At least a couple of different versions for *DEL* are carefully presented in [4].

[3]More about information dynamics and the dynamic turn can be found in [1].

[4]The mistake consisted in a purportedly valid reduction schema involving quantifiers and

tization had been proposed for quantified *PAL* before Kishida's paper (even considering simple models with constant domain, he insists).

On the other hand, Kishida's proposal [12] had its own peculiarities. Besides his elegant, but complicated, non standard semantic treatment (a combination of neighborhood and sheaf semantics), comitted to a (philosophically controversial) counterpart theory of individuals, his approach also had two important limitations: single-agent scenarios and *closed announcements* (i.e., announcements whose contents have no free individual variables).

Our contribution tries to fill this gap. We provide a family of axiom systems for *FOPAL* considering multi-agent scenarios, standard relational (Kripkean style) semantics, usual quantification domains (without individual counterparts), and no restrictions on the individual variables in the contents of announcements.

Moreover, we are interested in *complete* systems, and this reasonable choice has taken its toll, which requires an explanation. Usually, *PAL* systems include epistemic operators for *common knowledge*, due to interesting relations between public announcements and this peculiar collective epistemic state. However, it is known that even weak versions of (static) first-order epistemic logic with common knowledge are not axiomatizable [19], which forces us to restrict our present focus on *PAL without* common knowledge modalities.

Also, for philosophical reasons, our approach has opted for actualist quantification, as well as a relational semantic with variable domains. In order to implement these preferences, we resort to a *free logic* axiomatization, instead of a classical quantificational basis. Our guidelines are strongly influenced by well-known discussions on first-order modal logics and variable domain models — see, for example, [3, ch.15] [5, ch.4] [2, ch.9].

2 Syntax

Consider a non-empty set $A = \{i_1, \ldots, i_n\}$ with n epistemic agents ($n \in \mathbb{N}$). A *FOPAL language* $\mathcal{L}^n_{\mathcal{K}[.]}$ contains these lists of primitive symbols:

1. individual variables $x_0, x_1, \ldots$;
2. n-ary predicates $P^n_0, P^n_1, \ldots$;
3. propositional constant $\bot$;
4. identity symbol $=$;

announcement operators, which in fact resulted invalid w.r.t. the standard semantic approach.

5. a Boolean operator $\rightarrow$;
6. universal (actualist) quantifier $\forall$;
7. epistemic modal operators $\mathcal{K}_i$ $(i \in A)$;
8. public announcement operators [.].

Before we define a syntax for $\mathcal{L}^n_{\mathcal{K}[.]}$, a remark is needed. The reader may have noticed that the lists above don't include symbols for individual constants. The adoption of flexible individual terms requires much more semantic difficulties, and might easily produce incomplete systems [6, p.289-302][3, p.335-342]. In particular, the inclusion of non-rigid individual constants demands a considerable complication in completeness proofs, and won't guarantee complete first-order systems weaker than S5.

Since we want to give a family of complete systems for *FOPAL* capable of modelling contingent identities in natural language (even if our identity symbol $=$ doesn't stand for a contingent identity), we'll leave out symbols for individual constants. Because of this, our formal treatment for contingent identities will be done in a indirect way. Whenever we want to denote objects through descriptions, the chosen strategy will be using equivalences between properties satisfiable only by a unique individual in each model point.[5]

Definition 2.1 ($\mathcal{L}^n_{\mathcal{K}[.]}$ formulas) *The formulas in $\mathcal{L}^n_{\mathcal{K}[.]}$ are defined by this grammar, in BNF notation:*

$\varphi :: \bot \mid P^k(\vec{x}) \mid (x = y) \mid \varphi \rightarrow \varphi \mid \forall x\varphi \mid \mathcal{K}_i\varphi \mid [\varphi]\varphi$

We consider as *atomic formulas* only $\bot$ and instances of schemas $P^k(\vec{x})$ and $(x = y)$. Lowercase Greek letters φ, ψ, ..., will be metavariables for any formulas; lowercase Latin letters p, q, ..., are metavariables for atomic formulas only. The usual abbreviations for formulas with symbols $\neg$, $\neq$, $\wedge$, $\vee$, $\leftrightarrow$, $\exists$ are assumed here, and we'll mention explicitly only the following:

Definition 2.2 ($\mathcal{L}^n_{\mathcal{K}[.]}$ abbreviations) *Consider the following abbreviations for $\mathcal{L}^n_{\mathcal{K}[.]}$ formulas:*

1. $\mathsf{E}(x) =_{def} \exists y(y = x)$ *(where y is any variable different from x);*
2. $\langle\varphi\rangle\,\psi =_{def} \neg[\varphi]\neg\psi$;

[5] For space limitations, details on this particular topic aren't included in this paper, but can be found in [16, p.130-135].

We'll assume the usual conventions for free and bound (occurrences of) variables. Moreover, let's informally define a free individual occurrence of variable x in a formula φ as *replaceable by another variable* y iff y doesn't occur in a subformula $\forall y\psi$ in φ. The schema $\varphi(x/y)$ stands for the result of replacing all free occurrences of x in φ, if there's any, for occurrences of y, wherever x is replaceable for y in φ.

3 Semantics

After the basic syntax for $\mathcal{L}^n_{\mathcal{K}[.]}$, let's describe our relational semantics.

Definition 3.1 (Epistemic frame) *Let A be a non-empty finite set of epistemic agents. An* epistemic frame *F for $\mathcal{L}^n_{\mathcal{K}[.]}$ is a tuple $(W, \{R_i\}_{i\in A})$, such that W is a non-empty set (of evaluation points) and each R_i is a (possibly empty) binary relation between members of W. Besides:*

1. *F is reflexive iff, for every $w \in W$ and each R_i: $(w, w) \in R_i$;*
2. *F is transitive iff, for any $w, w', w'' \in W$ and each R_i: $(w, w') \in R_i$ & $(w', w'') \in R_i \Rightarrow (w, w'') \in R_i$;*
3. *F is euclidean iff, for any $w, w', w'' \in W$ and each R_i: $(w, w') \in R_i$ & $(w, w'') \in R_i \Rightarrow (w', w'') \in R_i$.*

Definition 3.2 (Augmented epistemic frame) *An* augmented epistemic frame *for $\mathcal{L}^n_{\mathcal{K}[.]}$ is a tuple $(W, \{R_i\}_{i\in A}, D)$, such that W and $\{R_i\}_{i\in A}$ are exactly as in an epistemic frame, and D is a non-empty set (of objects).*

Definition 3.3 (Epistemic model) *An* epistemic model *for $\mathcal{L}^n_{\mathcal{K}[.]}$ is a tuple $(W, \{R_i\}_{i\in A}, D, Q, I)$, such that W, $\{R_i\}_{i\in A}$ and D are exactly as in an augmented epistemic frame, Q is a function from W in 2^D, and I is an interpretation for $\mathcal{L}^n_{\mathcal{K}[.]}$ such that $I(P^k, w) \subseteq D^k$.*

Before we give the satisfiability conditions, as usual, we'll need to define what is a variant of an assignment of denotations to individual variables.

Definition 3.4 (Variant of an assignment) *Let $M = (W, \{R_i\}_{i\in A}, D, Q, I)$ be an epistemic model for $\mathcal{L}^n_{\mathcal{K}[.]}$, and let σ be an assignment of members of D to each individual variable x_0, $x_1, \ldots$ in $\mathcal{L}^n_{\mathcal{K}[.]}$. A* variant of σ w.r.t. a variable *x_j is an assignment $\sigma(x_j/o)$ exactly as σ except at most on its jth place, where $\sigma(x_j/o)(x_j) = o$, for some $o \in D$.*

Definition 3.5 (Satisfiability conditions) *Let $M = (W, \{R_i\}_{i \in A}, D, Q, I)$ be an epistemic model for $\mathcal{L}^n_{\mathcal{K}[.]}$. Also, let $w \in W$ and σ be an assignment of members of $Q(w)$ to the individual variables. We define the satisfiability relation $\models$ in the following manner:*

1. $(M^\sigma, w) \not\models \bot$;
2. $(M^\sigma, w) \models P^k(x_{i_1}, ..., x_{i_k})$ *iff* $(\sigma(x_{i_1}), \ldots, \sigma(x_{i_k})) \in I(P^k, w)$;
3. $(M^\sigma, w) \models (x = y)$ *iff* $\sigma(x) = \sigma(y)$;
4. $(M^\sigma, w) \models \varphi \to \psi$ *iff either* $(M^\sigma, w) \not\models \varphi$ *or* $(M^\sigma, w) \models \psi$;
5. $(M^\sigma, w) \models \forall x \varphi$ *iff, for every* $o \in Q(w)$, $(M^{\sigma(x/o)}, w) \models \varphi$;
6. $(M^\sigma, w) \models \mathcal{K}_i \varphi$ *iff, for every* $w' \in W$, $wR_iw' \Rightarrow (M^\sigma, w') \models \varphi$;
7. $(M^\sigma, w) \models [\varphi]\psi$ *iff, for every* (M', w'), *if* $M' = M|_{\varphi^\sigma}$ *and* $w' = w$, *then* $(M'^\sigma, w') \models \psi$;

where $M|_{\varphi^\sigma} = (W^!, \{R_i\}^!_{i \in A}, D^!, Q^!, I^!)$ *is an epistemic model such that:*

1. $W^! = \|\varphi\|^\sigma_M = \{w \in W : (M^\sigma, w) \models \varphi\}$;
2. $D^! = D$ *and, for each* $w \in W^!$, $Q^!(w) = Q(w)$;
3. *each assignment* τ *over* $M|_{\varphi^\sigma}$ *behaves exactly as its homonym on* M;
4. $R^!_i = R_i \cap (W^! \times W^!)$;
5. *For each* $w \in W^!$, $I^!(P^k, w) = I(P^k, w)$ *and* $I^!(\bot, w) = I(\bot, w)$.

By convenience, let's refer to $M|_{\varphi^\sigma}$ *as an* update of model M w.r.t. φ and σ.

Note that, although the set W of worlds (evaluation points) in a model M may be modified after an update, changing the underlying frame for M, the initial domain D of M remains the same in the updated model. Consequently, an assignment of objects to individual variables isn't affected by an update. Besides, the indication of the assignment σ in notation $\|\varphi\|^\sigma_M$ allow us to track, so to speak, the original assignment σ, if we are led to work with its variants in the updated model.

It might be useful to detail the truth condition for formulas containing the dual modality $\langle . \rangle$, which appears in the next corollary, and whose proof is omitted due to its simplicity.

Corollary 3.6 *From Definitions 2.2 and 3.5, it follows the truth condition:* $(M^\sigma, w) \vDash \langle\varphi\rangle\, \psi$ *iff there exists an epistemic model* (M', w') *such that* $M' = M|_{\varphi^\sigma}$ *and* $w' = w$ *and* $(M'^\sigma, w') \vDash \psi$.

For reasons of space, we won't discuss or prove many semantic details and metaproperties for *FOPAL* that are strongly similar to *PAL*, including most schemas in next metatheorems, which are stated here only for clarity purposes in further proofs.[6] Let's focus on *FOPAL* specificities.

Theorem 3.7 *Consider an arbitrary epistemic model* M*, and a point* $w \in W$ *in* M*, and an assignment* σ *over* M*. For any formulas* φ *and* ψ *in* $\mathcal{L}^n_{\mathcal{K}[.]}$*, as well as any atomic formula* p *in that language, we have that:*

1. *if* $(M^\sigma, w) \vDash \langle\varphi\rangle\psi$*, then* $(M^\sigma, w) \vDash \varphi$*;*
2. *if* $(M^\sigma, w) \nvDash [\varphi]\psi$*, then* $(M^\sigma, w) \vDash \varphi$*;*
3. *if* $(M^\sigma, w) \vDash \varphi$*, then* $(M^\sigma, w) \vDash \langle\varphi\rangle\top$*;*
4. *if* $(M^\sigma, w) \nvDash \varphi$*, then* $(M^\sigma, w) \vDash [\varphi]\psi$*;*
5. *if* $(M|^\sigma_{\varphi^\sigma}, w) \vDash p$*, then* $(M^\sigma, w) \vDash p$*;*
6. *if* $(M|^\sigma_{\varphi^\sigma}, w) \nvDash p$*, then* $(M^\sigma, w) \nvDash p$*;*
7. $(M|^\sigma_{\varphi^\sigma}, w) \vDash p$ *iff* $(M^\sigma, w) \vDash p$ *(whenever there is a point* $(M|_{\varphi^\sigma}, w)$*).*

Theorem 3.8 *Let* φ*,* ψ *and* ξ *be any formulas, and* p *be any atomic formula, in* $\mathcal{L}^n_{\mathcal{K}[.]}$*. We have the following properties:*

1. $\vDash \langle\varphi\rangle\psi \to [\varphi]\psi$ *(public announcements are functional)*
2. $\nvDash \langle\varphi\rangle\top$ *(public announcements are partial)*
3. $\vDash [p]p$ *(public announcements preserve atomic formulas)*

Theorem 3.9 *Let* φ*,* ψ *and* ξ *be any formulas, and* p *be any atomic formula, in* $\mathcal{L}^n_{\mathcal{K}[.]}$*. The following schemas are valid:*

1. $\vDash [\varphi]p \leftrightarrow (\varphi \to p)$
2. $\vDash [\varphi](\psi \to \xi) \leftrightarrow ([\varphi]\psi \to [\varphi]\xi)$

[6] The reader might want to check those proofs and comments concerning *PAL* in references already mentioned in this paper, e.g. [4, ch.4]. Detailed proofs and comments on each schema in *FOPAL* context, can be found in [16].

3. $\vDash [\varphi]\mathcal{K}_i\psi \leftrightarrow (\varphi \rightarrow \mathcal{K}_i[\varphi]\psi)$

4. $\vDash [\varphi][\psi]\xi \leftrightarrow [(\varphi \wedge [\varphi]\psi)]\xi$

5. $\vDash (x = y) \rightarrow [\varphi](x = y)$

6. $\vDash (x \neq y) \rightarrow [\varphi](x \neq y)$

7. $\vDash \psi \quad \Rightarrow \quad \vDash [\varphi]\psi$

Proof. We'll prove only schema *5*. Suppose, for an arbitrary epistemic model M, an assignment σ and an evaluation point $w \in W$ in M, that $(M^\sigma, w) \vDash (x = y)$. Also, consider an arbitrary formula φ in $\mathcal{L}^n_{\mathcal{K}[.]}$. If $(M^\sigma, w) \nvDash \varphi$, by item *4* in Theorem 3.7, we know that $(M^\sigma, w) \vDash [\varphi](x = y)$. If $(M^\sigma, w) \vDash \varphi$, by item *3* in same theorem, we know that $(M^\sigma, w) \vDash \langle\varphi\rangle\top$; and, consequently, that there is some (M', w') such that $M' = M|_{\varphi^\sigma}$ and w' $= w$ and $(M'^\sigma, w') \vDash \top$. So, consider this model point $(M|^\sigma_{\varphi^\sigma}, w)$. As, by construction, the domain $D^!$ in the updated model is the same as in M, and so is the assignment σ; we can be certain that $(M|^\sigma_{\varphi^\sigma}, w) \vDash (x = y)$, since the object denoted by $\sigma(x)$ is the same as the one denoted by $\sigma(y)$ in M, and this is also the case in $M|_{\varphi^\sigma}$. Then, from $(M|^\sigma_{\varphi^\sigma}, w) \vDash (x = y)$, we can easily infer that $(M^\sigma, w) \vDash \langle\varphi\rangle(x = y)$, and, by item *1* of Theorem 3.8, that $(M^\sigma, w) \vDash [\varphi](x = y)$. By the way, schema *6* is proved in a very similar way, by supposing $(M^\sigma, w) \vDash (x \neq y)$. ■

Lemma 3.10 *Let φ and ψ be any formulas of $\mathcal{L}^n_{\mathcal{K}[.]}$, and let w be an arbitrary evaluation point in an epistemic model M, and let σ and τ be any assignments over domain D of M. Besides, suppose that some individual variable x doesn't occur free in φ. Then, for an arbitrary $o \in Q(w)$:*

1. $(M^\sigma, w) \vDash \varphi$ *iff* $(M^{\sigma(x/o)}, w) \vDash \varphi$ *;*

2. $\|\varphi\|^\sigma_M = \|\varphi\|^{\sigma(x/o)}_M$ *;*

3. $(M|^\tau_{\varphi^\sigma}, w) \vDash \psi$ *iff* $(M|^\tau_{\varphi^{\sigma(x/o)}}, w) \vDash \psi$ *.*

Proof. *1.* Assume that $(M^\sigma, w) \vDash \varphi$ and let $\sigma(x_j/o_j)$ be, as usual, exactly like σ except, at most, for x_j, where $\sigma(x_j/o_j)(x_j) = o_j$ (for some $o_j \in Q(w)$). Now, suppose that x_j doesn't occur free in φ. Then, we have two possibilities: either (i) subformulas φ' in φ don't include any occurrences of x_j, or (ii) x_j occur only in subformulas $\forall x_j\varphi'$ in φ. In case (i), the object assigned by σ to x_j is irrelevant, and $(M^\sigma, w) \vDash \varphi'$ always coincide with $(M^{\sigma(x_j/o_j)}, w) \vDash \varphi'$. In case (ii), it's enough to show that $(M^\sigma, w) \vDash \forall x_j \varphi'$ always coincide with $(M^{\sigma(x_j/o_j)}, w) \vDash \forall x_j \varphi'$. That's easy to see. Recall that $(M^\sigma, w) \vDash \forall x_j \varphi'$ ammounts to say that, for every $o \in Q(w)$, $(M^{\sigma(x_j/o)}, w) \vDash \varphi'$, and $(M^{\sigma(x_j/o_j)}, w) \vDash \forall x_j \varphi'$, to say that for every $o \in Q(w)$, $(M^{\sigma(x_j/o_j)(x_j/o)}, w) \vDash \varphi'$. Well, assignments σ and $\sigma(x_j/o_j)$ agree in each variable, except at most to x_j, and both conditions above entail that φ' is satisfied in point w of M by any $o \in Q(w)$ that replaces the denotations of both $\sigma(x_j)$ and $\sigma(x_j/o_j)(x_j)$, which are the only place where these denotations might have been different. Then, $(M^\sigma, w) \vDash \forall x_j \varphi'$ iff $(M^{\sigma(x_j/o_j)}, w) \vDash \forall x_j \varphi'$.

As $(M^\sigma, w) \vDash \varphi$ is totally determined by satisfiability of subformulas φ'; by showing that, for each subformula φ' of φ, $(M^\sigma, w) \vDash \varphi'$ coincides with $(M^{\sigma(x_j/o_j)}, w) \vDash \varphi'$, we are showing that $(M^\sigma, w) \vDash \varphi$ also coincides with $(M^{\sigma(x_j/o_j)}, w) \vDash \varphi$. As x_j and o_j were arbitrary too, item *1* of this lemma is guaranteed. (The other direction is obvious.)

2. As w was arbitrary in the proof for previous item, item *2* follows easily from definitions for $\|\varphi\|^\sigma_M$ and $\|\varphi\|^{\sigma(x/o)}_M$.

3. Trivial, based on previous items, and as soon as it's realized that $M|_{\varphi^\sigma}$ and $M|_{\varphi^{\sigma(x/o)}}$ are exactly the same model. ■

Next, we have a very important result, which should grant us dealing with what might be called *open announcements* — that is, announcement operators containing any first-order formula of $\mathcal{L}^n_{\mathcal{K}[.]}$, including those with free variables.

Theorem 3.11 (Barcan-like schema for announcements) *Let φ and ψ be any formulas of $\mathcal{L}^n_{\mathcal{K}[.]}$. We have the following validity:*

$$\vDash [\varphi]\forall x\psi \;\leftrightarrow\; \forall y[\varphi]\psi(x/y) \text{ (where } y \text{ doesn't occur free in } \varphi\text{, neither in } \psi)$$

Proof. $(\Rightarrow)$ Suppose that, for arbitrary evaluation point (M, w) and assignment σ, that $(M^\sigma, w) \nvDash \forall y[\varphi]\psi(x/y)$ (where y doesn't occur free in φ, neither in ψ). Applying simple reasonings, we have that $(M^\sigma, w) \vDash \exists y\langle\varphi\rangle\neg\psi(x/y)$; that is, for some $o \in Q(w)$, it happens that $(M^{\sigma(y/o)}, w) \vDash \langle\varphi\rangle\neg\psi(x/y)$. By Corollary 3.6, there is some (M', w') such that $M' = M|_{\varphi^{\sigma(y/o)}}$ and $w' = w$ and $(M'^{\sigma(y/o)}, w') \vDash \neg\psi(x/y)$. More explicitly: for some $o \in Q(w)$, $(M|^{\sigma(y/o)}_{\varphi^{\sigma(y/o)}}, w) \vDash \neg\psi(x/y)$.

By construction, we know that $Q^!(w) = Q(w)$; consequently, it's safe to say that $(M|^\sigma_{\varphi^{\sigma(y/o)}}, w) \vDash \exists y\neg\psi(x/y)$. As, by hypothesis, φ doesn't contain any free occurrences of y, applying item *3* of lema 3.10, we know that $(M|^\sigma_{\varphi^\sigma}, w) \vDash \exists y\neg\psi(x/y)$. So, there is some (M', w') such that $M' = M|_{\varphi^\sigma}$ and $w' = w$ and $(M'^\sigma, w') \vDash \exists y\neg\psi(x/y)$; which, by definition, gives us $(M^\sigma, w) \vDash \langle\varphi\rangle\exists y\neg\psi(x/y)$; and, finally, $(M^\sigma, w) \nvDash [\varphi]\forall y\psi(x/y)$. It's easy to see that

$\forall y\psi(x/y)$, in aforementioned conditions, is simply an alphabetical variant of $\forall x\psi$, and it follows that $(M^\sigma, w) \nvDash [\varphi]\forall x\psi$.

($\Leftarrow$) Suppose, now, that $(M^\sigma, w) \nvDash [\varphi]\forall x\psi$. By usual reasonings, we have $(M^\sigma, w) \vDash \langle\varphi\rangle\exists x\neg\psi$. By Corollary 3.6, we also have that, for some (M', w'), $M' = M|_{\varphi^\sigma}$ and $w' = w$ and $(M'^\sigma, w') \vDash \exists x\neg\psi$. Consider an arbitrary individual variable y without free occurrences in φ, neither in ψ. Well, it's easy to see that $\exists x\neg\psi$ is just an alphabetical variant of $\exists y\neg\psi(x/y)$, and, then, $(M'^\sigma, w') \vDash \exists y\neg\psi(x/y)$. Thus, for some $o \in Q^!(w')$, $(M'^{\sigma(y/o)}, w') \vDash \neg\psi(x/y)$. That is, for some $o \in Q^!(w')$, $(M|^{\sigma(y/o)}_{\varphi^\sigma}, w) \vDash \neg\psi(x/y)$.

By item *3* in Lemma 3.10, it's safe to say that, for some $o \in Q^!(w')$, $(M|^{\sigma(y/o)}_{\varphi^{\sigma(y/o)}}, w) \vDash \neg\psi(x/y)$; that is, for some (M', w'), $M' = M|_{\varphi^{\sigma(y/o)}}$ and $w' = w$ and $(M'^{\sigma(y/o)}, w') \vDash \neg\psi(x/y)$. By construction, we know that $Q^!(w) = Q(w)$. Applying Corollary 3.6, we have that, for some $o \in Q(w)$, $(M^{\sigma(y/o)}, w) \vDash \langle\varphi\rangle\neg\psi(x/y)$; then, $(M^\sigma, w) \vDash \exists y\langle\varphi\rangle\neg\psi(x/y)$. By usual reasonings, $(M^\sigma, w) \nvDash \forall y[\varphi]\psi(x/y)$. ■

If we informally define as *closed public announcements* those public announcement operators which contain only closed formulas (i.e., with no free variables), we can easily derive the following corollary as a particular case of previous theorem, because we no longer have to replace the bound variable by an alphabetical variant to guarantee the equivalence. Its proof is simple and will be omited.

Corollary 3.12 (Barcan schema for closed announcements) *Let φ be a closed formula, and ψ be any formula, of $\mathcal{L}^n_{\mathcal{K}[.]}$. We have the following validity:*

$\vDash [\varphi]\forall x\psi \leftrightarrow \forall x[\varphi]\psi$

It's important to show that the well-known property of substitution between equivalents within formulas works here, as long as we observe a simple restriction.

Definition 3.13 (Off-announcement occurrence of a formula) *Let's define, for any formulas φ, ψ and ξ in $\mathcal{L}^n_{\mathcal{K}[.]}$, that an occurrence of φ is an* off-announcement occurrence *whenever this occurrence isn't a subformula of some occurrence of ψ such that $[\psi]\xi$.*

Observe that an off-announcement occurrence can happen within the scope of an announcement, or it can even contain an announcement, but it can't be itself the *content* of the announcement. For example, the occurrence of φ in $[\psi]\varphi$, as well as the occurrence of $[\psi]\xi$ in $[\varphi][\psi]\xi$ are off-announcement occurrences, but, in the latter case, the occurrence of φ isn't an off-announcement one. Now, let's proceed to the substitution property.

Lemma 3.14 (Substitution of equivalents) *Let φ, ψ and ζ be any formulas in $\mathcal{L}^n_{\mathcal{K}[.]}$. By schema $\zeta(\varphi/\psi)$, we understand the result of replacing some (or all) off-announcement occurrences of φ, if there are any, in ζ, by occurrences of ψ. Then,*

$$\text{if } \vDash \varphi \leftrightarrow \psi, \text{ then, } \vDash \zeta \leftrightarrow \zeta(\varphi/\psi)$$

Proof. By induction on ζ. *I.h.*: the lemma applies to ζ with length $< n$. We'll prove only the relevant situations, i.e., when ζ contains off-announcements occurrences of φ. The remaining situations are proved as in literature on (static) first-order epistemic logic — e.g., [16, p.74].

ζ has the form $\zeta' \to \zeta''$. Obviously, any occurrences of φ should happen either in ζ', or in ζ''. In any case, by i.h., we know that $\zeta' \leftrightarrow \zeta'(\varphi/\psi)$ and $\zeta'' \leftrightarrow \zeta''(\varphi/\psi)$. By propositional reasoning, we obtain $(\zeta' \to \zeta'') \leftrightarrow (\zeta'(\varphi/\psi) \to \zeta''(\varphi/\psi))$, which, obviously, is the same as $(\zeta' \to \zeta'') \leftrightarrow (\zeta' \to \zeta'')(\varphi/\psi)$.

ζ has the form $\forall x\zeta'$. Of course, any occurrences of φ will be subformulas of ζ'. From tautology $\zeta' \leftrightarrow \zeta'$, we have, by i.h., $\zeta' \leftrightarrow \zeta'(\varphi/\psi)$. By usual first-order reasonings, we also have $\forall x\zeta' \leftrightarrow \forall x\zeta'(\varphi/\psi)$, which ammounts to say that $\forall x\zeta' \leftrightarrow (\forall x\zeta')(\varphi/\psi)$.

ζ has the form $\mathcal{K}_i\zeta'$. Obviously, any occurrences of φ should be in ζ'. From tautology $\zeta' \leftrightarrow \zeta'$, we infer, by i.h., $\zeta' \leftrightarrow \zeta'(\varphi/\psi)$. By usual modal reasonings, we obtain $\mathcal{K}_i\zeta' \leftrightarrow \mathcal{K}_i\zeta'(\varphi/\psi)$, which is the same as $\mathcal{K}_i\zeta' \leftrightarrow (\mathcal{K}_i\zeta')(\varphi/\psi)$.

ζ has the form $[\zeta']\zeta''$. As we are considering only off-announcements occurrences of φ, they should occur as subformulas of ζ''. Then, again from tautology $\zeta'' \leftrightarrow \zeta''$, we obtain, by i.h., $\zeta'' \leftrightarrow \zeta''(\varphi/\psi)$. By items *7* and *2* in Theorem 3.9, it's easy to see that $[\zeta']\zeta'' \leftrightarrow [\zeta']\zeta''(\varphi/\psi)$; which ammounts to say that $[\zeta']\zeta'' \leftrightarrow ([\zeta']\zeta'')(\varphi/\psi)$.[7] ∎

4 Axiomatizations and completeness issues

Now, we provide a basic axiom system $\mathrm{QK}^n_{[.]}$ for *FOPAL*. As usual, $\Rightarrow$ stands for the inference of formulas in *FOPAL*, $\vdash \varphi$ means that φ is a theorem, and we say that a formula φ is derivable from a set Γ of formulas (notation: $\Gamma \vdash \varphi$) iff, for some $\gamma_1, \gamma_2, ..., \gamma_n \in \Gamma$, it is the case that $\vdash (\gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_n) \to \varphi$.

[7] Actually, we'd have to prove first, from item *2* in Theorem 3.9, that $\vDash [\varphi](\psi \leftrightarrow \xi) \leftrightarrow ([\varphi]\psi \leftrightarrow [\varphi]\xi)$; but, this is straightforward.

PL	*(all classical tautologies)*
MP	$\varphi \to \psi,\ \varphi \Rightarrow \psi$
K	$\mathcal{K}_i(\varphi \to \psi) \to (\mathcal{K}_i\varphi \to \mathcal{K}_i\psi)$
Nec	$\varphi \Rightarrow \mathcal{K}_i\varphi$
Atomic$[\cdot]$	$[\varphi]p \leftrightarrow (\varphi \to p)$
Distribution$[\cdot]$	$[\varphi](\psi \to \xi) \leftrightarrow ([\varphi]\psi \to [\varphi]\xi)$
Knowledge$[\cdot]$	$[\varphi]\mathcal{K}_i\psi \leftrightarrow (\varphi \to \mathcal{K}_i[\varphi]\psi)$
Composition$[\cdot]$	$[\varphi][\psi]\xi \leftrightarrow [(\varphi \wedge [\varphi]\psi)]\xi$
Barcan$[\cdot]$	$[\varphi]\forall x\psi \leftrightarrow \forall y[\varphi]\psi(x/\ y)$ (when y isn't free in φ, nor ψ)
Nec$[\cdot]$	$\varphi \Rightarrow [\psi]\varphi$
Vac$\forall$	$\forall x\varphi \leftrightarrow \varphi$ (when x isn't free in φ)
Distr$\forall$	$\forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall x\psi)$
Inst$\forall$	$(\forall x\varphi \wedge \mathsf{E}(y)) \to \varphi(x/y))$
$\mathsf{E}\forall$	$\forall x\mathsf{E}(x)$
Gen$\forall$	$\varphi \Rightarrow \forall x\varphi$
Gen$\forall^n$	$\varphi_1 \to \mathcal{K}_i(\varphi_2 \to \ldots \mathcal{K}_i(\varphi_n \to \mathcal{K}_i\psi)\ldots)$ $\Rightarrow \varphi_1 \to \mathcal{K}_i(\varphi_2 \to \ldots \mathcal{K}_i(\varphi_n \to \mathcal{K}_i\forall x\psi)\ldots)$ (when x isn't free in $\varphi_1, \ldots, \varphi_n$)
Id	$(x = x)$
Subst	$(x = y) \to (\varphi \to \varphi')$ (where φ' is as φ except for having free y in 0 or more places where φ contains free x)
NecDif	$(x \neq y) \to \mathcal{K}_i(x \neq y)$

Figure 1: Axiom system $\mathrm{QK}^n_{[.]}$

Definition 4.1 ($\mathbf{QK}^n_{[.]}$ and extensions) *The axiom schemas in Figure 1 define the first-order axiom system $QK^n_{[.]}$ for FOPAL. Besides, by adding combinations of the axiom schemas below, we define the respective axiom systems obtained based in $QK^n_{[.]}$:*

1. $\mathrm{QK}^n_{[.]}$ *plus* $\mathcal{K}_i\varphi \to \varphi$ *(veridicality):* $\mathrm{QKT}^n_{[.]}$
2. $\mathrm{QKT}^n_{[.]}$ *plus* $\mathcal{K}_i\varphi \to \mathcal{K}_i\mathcal{K}_i\varphi$ *(positive introspection):* $\mathrm{QKT4}^n_{[.]}$
3. $\mathrm{QKT}^n_{[.]}$ *plus* $\neg\mathcal{K}_i\varphi \to \mathcal{K}_i\neg\mathcal{K}_i\varphi$ *(negative introspection):* $\mathrm{QKT5}^n_{[.]}$.

Many interesting theorems, derived rules and properties could be discussed here, including soundness proofs; however, for limitations of space, we move forward to completeness issues.[8]

[8]The reader is advised to check detailed comments and proofs in [16, ch.6].

Concerning completeness of each of the systems $\mathrm{QK}^n_{[.]}$, $\mathrm{QKT}^n_{[.]}$, $\mathrm{QKT4}^n_{[.]}$ and $\mathrm{QKT5}^n_{[.]}$, we can resort to the standard strategy for *PAL*, which *reduces*, so to speak, each formula from $\mathcal{L}^n_{\mathcal{K}[.]}$ to a corresponding formula in the *static fragment* $\mathcal{L}^n_{\mathcal{K}}$, whose syntax is exactly as $\mathcal{L}^n_{\mathcal{K}[.]}$'s, except for the lack of public announcement operators and, obviously, formulas including these operators.

Thus, we would just have to determine a static first-order epistemic axiom system that would be complete w.r.t. some class of frames, and, by reducing our dynamic system to its static version, the former would be automatically complete w.r.t. the same class of frames.[9] By the way, this is the main reason for choosing precisely those axiom schemas labelled with $[\cdot]$ in our list on Figure 1, usually called "reduction axioms".

Now, in order to prove completeness for our *FOPAL* systems by means of a reduction to their static (complete) counterparts, we first need to define a static *translation* for every *FOPAL* formula, and then prove that each of them is equivalent to its (static) correspondent.

Definition 4.2 (Translation of a formula from $\mathcal{L}^n_{\mathcal{K}[.]}$) *Let φ, ψ and ξ be any formulas, and p be any atomic formula, both in $\mathcal{L}^n_{\mathcal{K}[.]}$ and in $\mathcal{L}^n_{\mathcal{K}}$. We define recursively the* translation of a formula from $\mathcal{L}^n_{\mathcal{K}[.]}$ into $\mathcal{L}^n_{\mathcal{K}}$ *as a mapping* $\mathbf{t}$*:* $\mathcal{L}^n_{\mathcal{K}[.]} \longrightarrow \mathcal{L}^n_{\mathcal{K}}$ *such that:*

1. $\mathbf{t}(p) = p$
2. $\mathbf{t}(\varphi \rightarrow \psi) = \mathbf{t}(\varphi) \rightarrow \mathbf{t}(\psi)$
3. $\mathbf{t}(\forall x\varphi) = \forall x\mathbf{t}(\varphi)$
4. $\mathbf{t}(\mathcal{K}_i\varphi) = \mathcal{K}_i\mathbf{t}(\varphi)$
5. $\mathbf{t}([\varphi]p) = \mathbf{t}(\varphi \rightarrow p)$
6. $\mathbf{t}([\varphi](\psi \rightarrow \xi)) = \mathbf{t}([\varphi]\psi \rightarrow [\varphi]\xi)$
7. $\mathbf{t}([\varphi]\mathcal{K}_i\psi) = \mathbf{t}(\varphi \rightarrow \mathcal{K}_i[\varphi]\psi)$
8. $\mathbf{t}([\varphi]\forall x\psi) = \mathbf{t}(\forall y[\varphi]\psi(x/\ y))$
   (where y is any individual variable not free in φ, nor in ψ)
9. $\mathbf{t}([\varphi][\psi]\xi) = \mathbf{t}([\varphi \wedge [\varphi]\psi]\xi)$

The next corollary will be useful in our proofs. Its proof should be easy enough and will be omitted here.

[9] Of course, completeness of *PAL* systems can be proved through other strategies. See also [18] for a discussion on different axiomatizations for propositional *PAL*.

Corollary 4.3 *The following conditions follow from the previous definition:*

1. $\mathbf{t}(\neg\varphi) = \neg\, \mathbf{t}(\varphi)$
2. $\mathbf{t}([\varphi]\neg\psi) = \mathbf{t}(\varphi \to \neg[\varphi]\psi)$
3. $\mathbf{t}(\varphi \wedge \psi) = \mathbf{t}(\varphi) \wedge \mathbf{t}(\psi)$

As it should be obvious by now, the mapping $\mathbf{t}$ is directly related to those aforementioned reduction axioms, and this fact reveals their main motivation: to enable a progressive migration of a public announcement operator within a dynamic formula, subformula by subformula, until it reaches the point where the scope of the announcement is just an atomic formula, when the whole formula can be replaced by a (completely) static correspondent (without announcement operators).

It might be useful making a brief note on item *8* in Definition 4.2, which is clearly related to axiom schema **Barcan**$[\cdot]$, which, by its turn, had its validity proved on Theorem 3.11. The restriction in **Barcan**$[\cdot]$ is needed because, in order to make $[\varphi]\forall x\psi$ entail $\forall x[\varphi]\psi$, no free occurrence of a variable in φ should become bound by adding the quantifier prefix $\forall x$, and, when making $\forall x[\varphi]\psi$ entail $[\varphi]\forall x\psi$, no bound occurrence of a variable in φ becomes free after narrowing the scope of $\forall x$. Any of these possibilities would affect the semantic conditions, preventing the equivalence between both sides of the biconditional.

The use of an alphabetical variant with that aforementioned restriction in choosing the variable y avoids this trouble. Also, it serves the purpose of reduction, because $\forall x\psi$ can be easily proven to be equivalent to $\forall y\psi(x/y)$ when y isn't free in ψ (see, for example, [16, p.75]). With the additional care on pre-existing occurrences of y in the content of an announcement operator, that equivalence solves the puzzle of moving the announcement operator without messing with our initial free and bound variables.

Lemma 4.4 *Let φ, ψ, ξ and ζ be any formulas in $\mathcal{L}^n_{\mathcal{K}[.]}$. By $\zeta(\varphi/\psi)$, we understand the result of replacing some (or all) off-announcements occurrences of φ, if there are any, in formula ζ, by occurrences of ψ. The following rule is derivable in $QK^n_{[.]}$:*

$$\vdash \varphi \leftrightarrow \psi \quad \Rightarrow \quad \vdash \zeta \leftrightarrow \zeta(\varphi/\psi)$$

Proof. Induction on ζ. Very similar to Lemma 3.14, now using tautologies and theorems of $\mathrm{QK}^n_{[.]}$. ■

We already know that all the reduction axioms are valid (Theorems 3.9 and 3.11), and that the substitution of equivalents preserves validity (Lemma

3.14), and is also a derived rule in $\mathrm{QK}^n_{[.]}$ (Lemma 4.4). Now, we might want to check if all those axioms serve adequately to the purpose of continually moving the announcement operator within a formula, until there remains only atomic formulas in its scope and this announcement operator can be safely erased.

Let's see how this works with an example, using only **Atomic**$[\cdot]$, **Barcan**$[\cdot]$ and substitution of equivalents.

Example 4.5 *Reduction of* $\forall x[P(x)]P(x) \to [P(x)]\forall x P(x)$*:*

1. $\forall x[P(x)]P(x) \to [P(x)]\forall x P(x)$
2. $\forall x[P(x)]P(x) \to \forall y[P(x)]P(y)$
3. $\forall x(P(x) \to P(x)) \to \forall y(P(x) \to P(y))$

As our goal is to prove the completeness of $\mathrm{QK}^n_{[.]}$ (and its extensions listed above) by reducing them to their corresponding complete static first-order epistemic systems, this reduction will be assured only if we can prove the equivalence between of each formula in $\mathcal{L}^n_{\mathcal{K}[.]}$ and its corresponding static version in $\mathcal{L}^n_{\mathcal{K}}$ through translation $\mathbf{t}$. For this, it wouldn't be enough to make the usual induction on length of formulas. To mention just one simple case, consider **Atomic**$[\cdot]$. An instance of schema $\varphi \to p$ isn't a subformula of $[\varphi]p$, nor vice-versa; and cases like this would prevent the application of inductive hypothesis to show the desired equivalence.

Then, the reduction strategy requires another convenient and reliable ordering of formulas, known as *complexity measure*, which doesn't depend *exclusively* on the relation among a formula and its subformulas, although considers this relation. Next, we provide a definition of this measure, adapted for *FOPAL*.

Definition 4.6 (Complexity of a formula) *Let φ and ψ be any formulas, and p be any atomic formula, in $\mathcal{L}^n_{\mathcal{K}[.]}$. The* complexity of a formula in $\mathcal{L}^n_{\mathcal{K}[.]}$ *is a mapping $\mathbf{c}$: $\mathcal{L}^n_{\mathcal{K}[.]} \longrightarrow \mathbb{N}$, defined recursively in this way:*

1. $\mathbf{c}(p) = 1$
2. $\mathbf{c}(\varphi \to \psi) = 1 + max(\mathbf{c}(\varphi), \mathbf{c}(\psi))$
3. $\mathbf{c}(\mathcal{K}_i\varphi) = 1 + \mathbf{c}(\varphi)$
4. $\mathbf{c}(\forall x\varphi) = 1 + \mathbf{c}(\varphi)$
5. $\mathbf{c}([\varphi]\psi) = (2 + \mathbf{c}(\varphi))\cdot\, \mathbf{c}(\psi)$

Corollary 4.7 *Considering Definition 4.6, we have that:*

1. $\mathbf{c}(\neg\varphi) = 1 + \mathbf{c}(\varphi)$

2. $\mathbf{c}(\varphi \wedge \psi) = 2 + max(\mathbf{c}(\varphi), 1 + \mathbf{c}(\psi))$

Proof. *1.* Consider $\mathbf{c}(\neg\varphi)$. The unabbreviated formula would be $\mathbf{c}(\varphi \to \bot)$, which, by definition, has the complexity $1 + max(\mathbf{c}(\varphi), \mathbf{c}(\bot))$, and, as $\bot$ is atomic, $1 + max(\mathbf{c}(\varphi), 1)$. Now, if φ is atomic too, $\mathbf{c}(\varphi) = \mathbf{c}(\bot) = 1$. If φ isn't atomic, certainly $\mathbf{c}(\varphi) > \mathbf{c}(\bot)$. It's clear that we can always disregard $\mathbf{c}(\bot)$ in $max(\mathbf{c}(\varphi), \mathbf{c}(\bot))$; then, the initial expression is the same as $1 + \mathbf{c}(\varphi)$.

2. Consider $\mathbf{c}(\varphi \wedge \psi)$. The unabbreviated conjunctive formula gives us $\mathbf{c}(\neg(\varphi \to \neg\psi))$, which, by previous item in this proof, is the same measure as $1 + \mathbf{c}(\varphi \to \neg\psi)$. This complexity is equivalent to $1 + (1 + max(\mathbf{c}(\varphi), \mathbf{c}(\neg\psi)))$, that is, $2 + max(\mathbf{c}(\varphi), \mathbf{c}(\neg\psi))$. Now, again by previous item, that's the same as $2 + max(\mathbf{c}(\varphi), 1 + \mathbf{c}(\psi))$. ■

The next lemma makes sure that our function $\mathbf{c}$ truly imposes a rigorous ordering on the formulas of $\mathcal{L}^n_{\mathcal{K}[.]}$, by satisfying some desired properties. Besides, during its proof, it becomes clear why choosing the numerical constant 2 in item *5* of Definition 4.6. In fact, this is the least natural number that satisfies the lemma.[10]

Lemma 4.8 *Let φ and ψ be any formulas, and p be any atomic formula, in $\mathcal{L}^n_{\mathcal{K}[.]}$, and let $sub(\varphi)$ be the set of subformulas of φ. Then, we have:*

1. *if $\varphi \in sub(\psi)$, then, $\mathbf{c}(\varphi) \leq \mathbf{c}(\psi)$*

2. $\mathbf{c}(\varphi \to p) < \mathbf{c}([\varphi]p)$

3. $\mathbf{c}([\varphi]\psi \to [\varphi]\xi) < \mathbf{c}([\varphi](\psi \to \xi))$

4. $\mathbf{c}(\mathcal{K}_i[\varphi]\psi) < \mathbf{c}([\varphi]\mathcal{K}_i\psi)$

5. $\mathbf{c}(\forall x[\varphi]\psi) < \mathbf{c}([\varphi]\forall x\psi)$

6. $\mathbf{c}([\varphi \wedge [\varphi]\psi]\xi) < \mathbf{c}([\varphi][\psi]\xi)$

Proof. *1.* Induction on ψ. *(i)* ψ is atomic. Obviously, $\mathbf{c}(\psi) \leq \mathbf{c}(\psi)$.

I.h.: For $\mathbf{c}(\psi) = n$ (arbitrary), if $\varphi \in sub(\psi)$, then $\mathbf{c}(\varphi) \leq \mathbf{c}(\psi)$. Suppose, in each step below, that $\varphi \in sub(\psi)$. (Subcases where φ is ψ are obvious.)

[10] As a curiosity, the reader might want to compare our choice of number 2 with the alternative choice of number 4 in a similar context made in [4, p.188]. The reason has to do with our chosen primitive operators.

(ii) ψ is $\psi' \to \psi''$: Clearly, either $\varphi \in sub(\psi')$, or $\varphi \in sub(\psi'')$. By i.h., either $\mathbf{c}(\varphi) \leq \mathbf{c}(\psi')$, or $\mathbf{c}(\varphi) \leq \mathbf{c}(\psi'')$. By Definition 4.6, $\mathbf{c}(\psi' \to \psi'') = 1 + max(\mathbf{c}(\psi'), \mathbf{c}(\psi''))$. It's easy to see that, in any case, $\mathbf{c}(\varphi) \leq \mathbf{c}(\psi)$.

(iii) ψ is $\forall x\psi'$: Of course, $\varphi \in sub(\psi')$. By i.h., $\mathbf{c}(\varphi) \leq \mathbf{c}(\psi')$; and, as $\mathbf{c}(\forall x\psi') = 1 + \mathbf{c}(\psi')$, we have $\mathbf{c}(\varphi) \leq \mathbf{c}(\forall x\psi')$.

(iv) ψ is $\mathcal{K}_i\psi'$: Like above.

(v) ψ is $[\psi']\psi''$: Either $\varphi \in sub(\psi')$, or $\varphi \in sub(\psi'')$. By i.h., either $\mathbf{c}(\varphi) \leq \mathbf{c}(\psi')$, or $\mathbf{c}(\varphi) \leq \mathbf{c}(\psi'')$. By Definition 4.6, $\mathbf{c}([\psi']\psi'') = (2 + \mathbf{c}(\psi'))\cdot\, \mathbf{c}(\psi'')$. It's clear that, in any case, $\mathbf{c}(\varphi) \leq \mathbf{c}([\psi']\psi'')$.

2. By definition, $\mathbf{c}(\varphi \to p) = 1 + max(\mathbf{c}(\varphi), \mathbf{c}(p))$
$= 1 + max(\mathbf{c}(\varphi), 1)$
$= 1 + \mathbf{c}(\varphi)$ (because $\mathbf{c}(\varphi) \geq 1$).
On the other hand, $\mathbf{c}([\varphi]p) = (2 + \mathbf{c}(\varphi))\cdot\, \mathbf{c}(p)$
$= (2 + \mathbf{c}(\varphi)) \cdot 1$
$= 2 + \mathbf{c}(\varphi)$
Obviously, $1 + \mathbf{c}(\varphi) < 2 + \mathbf{c}(\varphi)$.

3. Without loss of generality, let $\mathbf{c}(\psi) \leq \mathbf{c}(\xi)$.
So, $\mathbf{c}([\varphi]\psi \to [\varphi]\xi) = 1 + max(\mathbf{c}([\varphi]\psi), \mathbf{c}([\varphi]\xi))$
$= 1 + max((2{+}\mathbf{c}(\varphi))\cdot\, \mathbf{c}(\psi), (2{+}\mathbf{c}(\varphi))\cdot\, \mathbf{c}(\xi))$
$= 1 + (2{+}\mathbf{c}(\varphi))\cdot\, \mathbf{c}(\xi)$
$= 1 + 2\cdot\, \mathbf{c}(\xi) + \mathbf{c}(\varphi)\cdot\, \mathbf{c}(\xi)$.
On the other hand, $\mathbf{c}([\varphi](\psi \to \xi)) = (2 + \mathbf{c}(\varphi))\cdot\, \mathbf{c}(\psi \to \xi)$
$= (2 + \mathbf{c}(\varphi)) \cdot (1 + max(\mathbf{c}(\psi), \mathbf{c}(\xi)))$
$= (2 + \mathbf{c}(\varphi)) \cdot (1 + \mathbf{c}(\xi))$
$= 2 + \mathbf{c}(\varphi) + 2 \cdot \mathbf{c}(\xi) + \mathbf{c}(\varphi)\cdot\, \mathbf{c}(\xi)$.
At last, $1 + 2\cdot\, \mathbf{c}(\xi) + \mathbf{c}(\varphi)\cdot\, \mathbf{c}(\xi) < 2 + \mathbf{c}(\varphi) + 2\cdot\, \mathbf{c}(\xi) + \mathbf{c}(\varphi)\cdot\, \mathbf{c}(\xi)$.

4. By definition, $\mathbf{c}(\mathcal{K}_i[\varphi]\psi) = 1 + \mathbf{c}([\varphi]\psi)$
$= 1 + ((2 + \mathbf{c}(\varphi)) \cdot \mathbf{c}(\psi))$
$= 1 + 2 \cdot \mathbf{c}(\psi) + \mathbf{c}(\varphi)\cdot\, \mathbf{c}(\psi)$
On the other hand, $\mathbf{c}([\varphi]\mathcal{K}_i\psi) = (2{+}\, \mathbf{c}(\varphi))\cdot\, \mathbf{c}(\mathcal{K}_i\psi)$
$= (2{+}\, \mathbf{c}(\varphi))\cdot\, (1{+}\, \mathbf{c}(\psi))$
$= 2{+}\, \mathbf{c}(\varphi) + 2\cdot\, \mathbf{c}(\psi) + \mathbf{c}(\varphi)\cdot\, \mathbf{c}(\psi)$.
So, $1 + 2\cdot\, \mathbf{c}(\psi) + \mathbf{c}(\varphi)\cdot\, \mathbf{c}(\psi) < 2 + \mathbf{c}(\varphi) + 2\cdot\, \mathbf{c}(\psi) + \mathbf{c}(\varphi)\cdot\, \mathbf{c}(\psi)$.

5. Very similar to last item.

6. By definition, $\mathbf{c}([\varphi \wedge [\varphi]\chi]\psi) = (2 + \mathbf{c}(\varphi \wedge [\varphi]\chi)) \cdot \mathbf{c}(\psi)$
$= (2 + (2 + max(\mathbf{c}(\varphi), 1{+}\, \mathbf{c}([\varphi]\chi)))) \cdot \mathbf{c}(\psi)$
$= (4 + max(\mathbf{c}(\varphi), 1{+}\, \mathbf{c}([\varphi]\chi))) \cdot \mathbf{c}(\psi)$
$= (4 + max(\mathbf{c}(\varphi), 1 + ((2{+}\, \mathbf{c}(\varphi))\cdot\, \mathbf{c}(\chi))))\cdot\, \mathbf{c}(\psi)$
$= (4 + (1 + ((2{+}\, \mathbf{c}(\varphi))\cdot\, \mathbf{c}(\chi))))\cdot\, \mathbf{c}(\psi)$

$$= (4 + (1 + (2\cdot\ \mathbf{c}(\chi) + \mathbf{c}(\varphi)\cdot\ \mathbf{c}(\chi))))\cdot\ \mathbf{c}(\psi)$$
$$= (5 + 2\cdot\ \mathbf{c}(\chi) + \mathbf{c}(\varphi)\cdot\ \mathbf{c}(\chi))\cdot\ \mathbf{c}(\psi).$$

On the other hand, $\mathbf{c}([\varphi][\chi]\psi) = (2 + \mathbf{c}(\varphi))\cdot\ \mathbf{c}([\chi]\psi)$
$$= (2 + \mathbf{c}(\varphi))\cdot\ (2 + \mathbf{c}(\chi))\cdot\ \mathbf{c}(\psi)$$
$$= (4 + 2\cdot\ \mathbf{c}(\varphi) + 2\cdot\ \mathbf{c}(\chi) + \mathbf{c}(\varphi)\cdot\ \mathbf{c}(\chi))\cdot \mathbf{c}(\psi).$$

Of course: $(5 + 2\cdot\ \mathbf{c}(\chi) + \mathbf{c}(\varphi)\cdot\ \mathbf{c}(\chi))\cdot\ \mathbf{c}(\psi)$
$$< (4 + 2\cdot\ \mathbf{c}(\varphi) + 2\cdot\ \mathbf{c}(\chi) + \mathbf{c}(\varphi)\cdot\ \mathbf{c}(\chi))\cdot\ \mathbf{c}(\psi).$$

(Since, for sure, $\mathbf{c}(\varphi) \geq 1$, the result is guaranteed.) ■

The next lemma is the the final step before completeness.

Lemma 4.9 (Equivalence between a formula and its translation) *Let φ be any formula in $\mathcal{L}^n_{\mathcal{K}[.]}$. Then,*

$$\vdash \quad \varphi \quad \leftrightarrow \quad \mathbf{t}\,(\varphi)$$

Proof. Induction on *complexity measure* of φ. Atomic case is trivial. *I.h.:* For each φ such that $\mathbf{c}(\varphi) < n$: $\vdash \varphi \leftrightarrow \mathbf{t}(\varphi)$.

(i) φ is $\varphi' \to \varphi''$. Consider $\vdash (\varphi' \to \varphi'') \leftrightarrow (\varphi' \to \varphi'')$. Since φ' and φ'' are subformulas of φ, we know, by item *1* of Lemma 4.8, that i.h. applies, because $\mathbf{c}(\varphi') \leq \mathbf{c}(\varphi)$ and $\mathbf{c}(\varphi'') \leq \mathbf{c}(\varphi)$. It's easy to see that $\vdash (\varphi' \to \varphi'') \leftrightarrow (\mathbf{t}(\varphi') \to \mathbf{t}(\varphi''))$, which coincides, by Definition 4.2, with $\vdash (\varphi' \to \varphi'') \leftrightarrow \mathbf{t}(\varphi' \to \varphi'')$.

(ii) φ is $\mathcal{K}_i\varphi'$. Since φ' is subformula of φ, by Lemma 4.8 and i.h., we obtain $\vdash \varphi' \leftrightarrow \mathbf{t}(\varphi')$. Applying **Nec** and **K**, we have $\vdash \mathcal{K}_i\varphi' \leftrightarrow \mathcal{K}_i\ \mathbf{t}(\varphi')$. By definition, $\mathcal{K}_i\ \mathbf{t}(\varphi') = \mathbf{t}(\mathcal{K}_i\varphi')$; consequently, $\vdash \mathcal{K}_i\varphi' \leftrightarrow \mathbf{t}(\mathcal{K}_i\varphi')$.

(iii) φ $\forall x\varphi'$. Similar to above, but applying **Gen**$\forall$, **Distr**$\forall$ and **MP** on the result of inductive hypothesis. At last, applying Definition 4.2, from $\vdash \forall x\varphi' \leftrightarrow \forall x\mathbf{t}(\varphi')$, we obtain $\vdash \forall x\varphi' \leftrightarrow \mathbf{t}(\forall x\varphi')$.

We have five last cases, in which $\mathbf{c}(\varphi) > k$, not necessarily involving subformulas of φ:

(iv) φ is $[\varphi']p$. By **Atomic**$[\cdot]$, we know that $\vdash [\varphi']p \leftrightarrow (\varphi' \to p)$. By item *2* of Lemma 4.8 and i.h., we infer that $\vdash (\varphi' \to p) \leftrightarrow \mathbf{t}(\varphi' \to p)$ and, then, that $\vdash [\varphi']p \leftrightarrow \mathbf{t}(\varphi' \to p)$. Applying Definition 4.2, we obtain $\vdash [\varphi']p \leftrightarrow \mathbf{t}([\varphi']p)$.

(v) φ is $[\varphi'](\varphi'' \to \varphi''')$. Very similar to above, but using **Distribution**$[\cdot]$, item *3* of Lemma 4.8, and Definition 4.2.

(vi) φ is $[\varphi']\mathcal{K}_i\varphi''$. Very similar to above, but using **Knowledge**$[\cdot]$, item *4* of Lemma 4.8, and Definition 4.2.

(vii) φ is $[\varphi']\forall x\varphi''$. Very similar to above, but using **Barcan**$[\cdot]$, item *5* of Lemma 4.8, and Definition 4.2.

(viii) φ is $[\varphi'][\varphi'']\varphi'''$. Very similar to above, but using **Composition**$[\cdot]$, item *6* of Lemma 4.8, and Definition 4.2. ■

Completeness result for $\mathrm{QK}^n_{[.]}$ now follows straight from previous definitions and lemmas.

Theorem 4.10 (Completeness of $\mathbf{QK}^n_{[.]}$) *Let φ be any formula in $\mathcal{L}^n_{\mathcal{K}[.]}$ that is also theorem of $QK^n_{[.]}$. Then, φ is valid in the class of all epistemic frames. That is:*

$$\textit{If } \vDash \varphi, \textit{ then } \vdash \varphi.$$

Proof. Suppose $\vDash \varphi$. Since $\mathrm{QK}^n_{[.]}$ is sound, we have, by Lemma 4.9, that $\vDash \varphi \leftrightarrow \mathbf{t}(\varphi)$. Then, it's easy to see that $\vDash \mathbf{t}(\varphi)$. By Definition 4.2, $\mathbf{t}(\varphi)$ belongs to $\mathcal{L}^n_{\mathcal{K}}$ and doesn't contain any public announcement operators. We already know that we are dealing with exactly the same epistemic models used for QK^n (a K-style first-order static epistemic system), and that semantic conditions for formulas of $\mathcal{L}^n_{\mathcal{K}[.]}$ without announcement operators are exactly the same as $\mathcal{L}^n_{\mathcal{K}}$'s. Then, as QK^n can be proven complete w.r.t. the class of all epistemic frames,[11] and from step $\vDash \mathbf{t}(\varphi)$ above, we conclude that QK^n contains $\vdash \mathbf{t}(\varphi)$ as theorem. Recall that $\mathrm{QK}^n_{[.]}$ is clearly an extension of QK^n, and this entails that $\vdash \mathbf{t}(\varphi)$ is also theorem of $\mathrm{QK}^n_{[.]}$. At last, from $\vdash \varphi \leftrightarrow \mathbf{t}(\varphi)$ being theorem of $\mathrm{QK}^n_{[.]}$, we conclude that this is also the case with $\vdash \varphi$. ∎

For reasons of space, we won't prove the next theorem, but its proof is straightforward, applying the same strategy above, and changing only the relevant systems and classes of epistemic frames.

Theorem 4.11 (Completeness of $\mathbf{QKT}^n_{[.]}$, $\mathbf{QKT4}^n_{[.]}$ and $\mathbf{QKT5}^n_{[.]}$) *The axiom systems $QKT^n_{[.]}$, $QKT4^n_{[.]}$ and $QKT5^n_{[.]}$ for FOPAL are complete w.r.t., respectively, the class of all reflexive epistemic frames, the class of all reflexive transitive epistemic frames, and the class of all reflexive euclidean epistemic frames. In other words, in each of the cases above, for any formula φ in $\mathcal{L}^n_{\mathcal{K}[.]}$:*

$$\textit{If } \vDash \varphi, \textit{ then } \vdash \varphi.$$

5 Future research

As discussed in literature, reduction axioms as a strategy for completeness won't work for any formulation of propositional *PAL*. For example, if we add an epistemic operator for *common knowledge*, it can be proven that fragments of this extended language can't be successfully reduced to a static formulation [4, p.231]; i.e., the former language is more expressive than the latter, when

[11] See, for example, [16, ch.3].

they are interpreted in the same class of models. Of course, as explained in the introduction, if we wanted to include common knowledge operators in *FOPAL*, we'd have to sacrifice completeness; however, a desirable improvement would be providing other complete axiom systems with greater expressivity (even without common knowledge operators), whose completeness obviously couldn't rely on reduction strategies to static systems. Another important improvement would be the inclusion of individual constants, dealing with the corresponding complications in the proof of metatheorems.

References

[1] J. van Benthem. *Logical Dynamics of Information and Interaction.* Cambridge University Press, 2011. ISBN 9781139500463.

[2] W.A. Carnielli and C. Pizzi. *Modalities and Multimodalities*, volume 12 of *Logic, Epistemology, and the Unity of Science.* Springer-Verlag, 2008. ISBN 0792353358.

[3] G.E. Hughes and M.J. Cresswell. *A New Introduction to Modal Logic.* Routledge, 1996. ISBN 0415126002.

[4] H. van Ditmarsch; W. van der Hoek and B. Kooi. *Dynamic Epistemic Logic (Synthese Library).* Springer, 2007. ISBN 1402069081.

[5] M. Fitting and Richard L. Mendelsohn. *First-Order Modal Logic (Synthese Library).* Springer, 1998. ISBN 0792353358.

[6] J. W. Garson. Quantification in Modal Logic *Handbook of Philosophical Logic: Volume II: Extensions of Classical Logic* (p.249-307). Reidel, 1984. ISBN 9789400962613.

[7] J. Gerbrandy and W. Groeneveld. Reasoning about information change. *Journal of Logic, Language and Information*, 6(2):147169, 1997. ISSN 0925-8531.

[8] J. U. Hansen. A Hybrid Public Announcement Logic with Distributed Knowledge. *Electronic Notes in Theoretical Computer Science*, 273:33-50, 2011. ISSN 1571-0661.

[9] D. Harel; D. Kozen and J. Tiuryn. *Dynamic Logic (Foundations of Computing).* The MIT Press, 2000. ISBN 0262082896.

[10] W. Holliday; T. Hoshi and T. F. Icard, III. A Uniform Logic of Information Dynamics. *Advances in Modal Logic 9.* College Publications, 2012. ISBN 9781848900684.

[11] W. Holliday; T. Hoshi and T. F. Icard, III. Information dynamics and uniform substitution. *Synthese*, 190(1):31-55, 2013. ISSN 0039-7857. DOI: 10.1007/s11229-013-0278-0.

[12] K. Kishida. Public announcements under sheaves. In *New Frontiers in Artificial Intelligence (Lecture Notes in Computer Science)*, 7856:96–108, 2013. ISBN 9783642399305.

[13] M. Ma. Mathematics of Public Announcements. *Proceedings of the Third International Conference on Logic, Rationality, and Interaction (LORI'11)*. p.193–205. Springer-Verlag, 2011. ISBN 9783642241291.

[14] M. Ma; A. Palmigiano and M. Sadrzadeh. Algebraic semantics and model completeness for intuitionistic public announcement logic. In *Annals of Pure and Applied Logic*, 165(4):963–995, 2014. ISSN 0168-0072.

[15] J. Plaza. Logic of public communications. *Synthese*, 158:165–179, 2007. ISSN 0039-7857

[16] M. K. F. Pereira. *Extensões de primeira-ordem para a Lógica do Anúncio Público* (*First-order extensions for Public Announcement Logic*, in Portuguese). PhD thesis, Programa de Pós-Graduação em Filosofia, Universidade Federal de Santa Catarina (UFSC), Florianópolis, 2015.

[17] Y. Wang and T. Ågotnes. Public Announcement Logic with Distributed Knowledge. *Logic, Rationality, and Interaction (Lecture Notes in Computer Science)*, p.328-341. Springer Berlin Heidelberg, 2011. ISBN 9783642241291. DOI: 10.1007/978-3-642-24130-7-24.

[18] Y. Wang and Q. Cao. On axiomatizations of public announcement logic. *Synthese*, 190(1):103–134, 2013. ISSN 0039-7857.

[19] F. Wolter. First Order Common Knowledge Logics. *Studia Logica: An International Journal for Symbolic Logic*. 65(2):249-271, 2000. ISSN 0039-3215.

Marcio Kléos Freire Pereira
Department of Philosophy
Federal University of Maranhão (UFMA)
Avenida dos Portugueses 1966, CEP 65080-805, São Luís, MA, Brazil
E-mail: `marcio.kleos@ufma.br`