



Translating non Interpretable Theories

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Abstract

Interpretations are generally regarded as the formal representation of the concept of translation. We do not subscribe to this view. A translation method must indeed establish relative consistency or have some uniformity. These are requirements of a translation. Yet, one can both be more strict or more flexible than interpretations are. In this article, we will define a general scheme translation. It should incorporate interpretations but also be compatible with more flexible methods. By doing so, we want to account for methods that seem to imply a sense of translation but are not reducible to interpretations. The main example will be the relative consistent proof between ZF and NBG given by Novak (1950). Further, we will explore a way of combining interpretations. This should account for truth conditions discarded by interpretations in translated theories.

Introduction

Many philosophers and mathematicians believe that *interpretations* [7, p. 61] are the sole admissible concept of translation between first-order theories. However, in the article *On What counts as a Translation* [1], I defended that we should see all translation methods with suspicion. Instead of claiming that a privileged method is a translation, one should start from the questions: What a translation should do? What should it preserve?

I argued that a translation in a theory T_1 should preserve features of the translated theory T_2 so that reference relations of T_1 can emulate the reference relations of the T_2 . This requirement is hardly formalizable. It depends extensively on how flexible one is to admit that a particular emulation represents reference relations of the emulated theory. Often, a theory is interpretable into another and yet it is not possible to construct a translation if one is more demanding about how formulas are mapped.

This is the case for PA and ZF_{fin} .¹ Those theories are mutually interpretable and yet they are not bi-interpretable.² Although ZF_{fin} can interpret PA, it still cannot see the copy of itself in PA as an isomorphic copy (extra requirement of bi-interpretation). The mathematical community perceived this result as evidence for not regarding ZF_{fin} as the set-theoretic equivalent of PA. Instead, they added the axiom of hereditarily finite sets in ZF_{fin} (ZF_{t-fin}); Kaye and Wong [2] proved that the resulting theory is bi-interpretable with PA.

The above result shows that being more demanding may reveal more subtle aspects of reducibility. One may then understand the failure of bi-interpretation as evidence that the flexibility in mutual interpretations created the illusion of the ontological equivalence of PA and ZF_{fin} – while a more demanding notion of emulation shows that ZF_{fin} cannot reduce PA to its reference relations.

Does this mean we should regard the more demanding concept to be the true concept of translation? We believe not, it can still be the case that a more demanding type of translation cannot account for any reduction among the theories, i.e. it so inflexible that we can make no comparison for the theories in question.

In this article, we will investigate the case in which there is no interpretation between two theories and yet there is a way of comparing those two theories. Comparing such theories is not foreign: it is widely common to prove relative consistency between theories by assuming a model for the first and then using this model to build a model for the second. Some wrongly assume that this kind of construction implies the existence of an interpretation between the theories. This is, for instance, the case of the proof of relative consistency between ZF and NBG given by Novak [3]. Although by assuming a model for ZF one can prove there is a model for NBG, there is no interpretation of NBG in ZF. We will, therefore, present a way in which we can use Novak's proof to generate a sense of translation of NBG into ZF. From that, we work for a generalized version of this kind of translation.

1 The problem of model theoretical construction

As argued before, we want to give an account of Novak's construction for NBG as a translation. It seems to be an impossible task, since the model theoretical technique (when used necessarily) is infinitary in nature. It is not possible

¹ ZF_{fin} stands for Zermelo-Fraenkel set theory without the infinity axiom and with the negation of the axiom of infinity.

²We will define interpretations further in the text. For now, one need only to remember it is a commonly used method of translation. Also, bi-interpretations are a further requirements over mutual interpretations.

by standard techniques to reduce NBG to ZFC without recurring to actual infinity. And it seems senseless to talk about translations for which we only have infinitary descriptions. I assume therefore the task to provide a translation that accommodates this critique.

The key step will be to separate the description of the translation and the proof that the translation is sound. While the method of translation must be constructive, for it must present the translated formula/s in every case, the soundness for the same translation does not depend on any kind of performance.

Definition 1.1 *We devise*

1. *Translation mapping: the process for generating formulas that emulate the meaning of the original formula.*
2. *Soundness of translation: proof that a translation map preserves the meaning.*

The idea, thus, will be to push every infinitary aspect of the model theoretical reduction to the proof of soundness. Still, we should care about how we will produce the formulas. Our strategy will be to make the technique of interpretations more flexible.

Let's now review what it means to provide an interpretation between two theories:

Definition 1.2 *An interpretation J is a mapping $\mathcal{L}_1 \longrightarrow \mathcal{L}_2$ that:*

1. *Preserves boolean structure:*

$$(\alpha \wedge \beta)^J = \alpha^J \wedge \beta^J.$$

2. *Uniformly substitute predicates:*

For each predicate P in \mathcal{L}_1 , there is a formula α in \mathcal{L}_2 such that

$$(P(\bar{x}))^J = \alpha(\bar{x}).$$

3. *Uniformly bound quantifications:*

There is a formula correspondent to the universe of quantification U in \mathcal{L}_2

$$(\forall x(\alpha))^J = \forall x(U(x) \rightarrow \alpha^J).$$

This definition only states the structural requirements on the mapping between the languages. Now we need to impose that the mapping emulates reference relations of the theory it is interpreting:

Definition 1.3 *J is an interpretation of T_1 into T_2 if*

1. *J is an interpretation of \mathcal{L}_1 into \mathcal{L}_2 .*
2. *and for every $\alpha \in \mathcal{L}_1$*

$$\text{if } T_1 \vdash \alpha, \text{ then } T_2 \vdash \alpha^J$$

1.1 Flexibilizing interpretation demands

There are many ways in which one can flexibilize interpretations: one can refrain from preserving the boolean structure, the uniformity of predicates and/or uniformity of quantifiers. Also, we can refrain from preserving theoremicity as in definition 1.3. Instead of preserving theorems, we may require that there is a way for the translating theory to understand “true” sentences of the original theory as “true in the translation”. We therefore hope that the flexible version of interpretation satisfies:

T_2 see as true each formula brought to a structure comprehensible to T_2 that T_1 proves.³

We symbolize this statement:

1. $\alpha^{Tr(T_2)}$ denotes “the formula α brought to a structure comprehensible to T_2 ”.
2. $T_2 \vdash^s \alpha^{Tr(T_2)}$ denotes “see as true” according to how $Tr(T_2)$ was defined.

As a result, we have

Definition 1.4 (General scheme of interpretation) *The pair $\langle Tr(T_2), \vdash^s \rangle$ is a translation if, for every $\alpha \in \mathcal{L}_1$,*

$$T_1 \vdash \alpha \text{ implies } T_2 \vdash^s \alpha^{Tr(T_2)}.$$

Still, we need to have a connection of \vdash^s with the notion of provability. The reason for that is that we hold that translations should imply relative consistency between theories. For this purpose, we need only to require:

Definition 1.5 (Consistency Requirement) *If $T_2 \vdash^s \alpha^{Tr(T_2)} \wedge \neg \alpha^{Tr(T_2)}$, then T_2 is inconsistent.*

From this, it is easily verifiable that a translation implies relative consistency of the theories involved. If T_1 is inconsistent, then $T_1 \vdash \alpha \wedge \neg \alpha$ for some α ; then, from the translation, $T_2 \vdash^s \alpha^{Tr(T_2)} \wedge \neg \alpha^{Tr(T_2)}$ – which implies that T_2 is inconsistent.

³Note that we require that T_1 proves a formula. Although this may also be flexibilized, we keep this requirement because it is already sufficient for our purposes in this article.

2 Model-style translation

Dealing with interpretations, we substitute each predicate in the translated language by a single formula in the translating theory. Here, we refrain from this impediment, mapping each predicate to a number of formulas (which may be denumerable). The context of occurrence should be enough to determine which is/are the correct interpretation/s for the predicate/s. This is an account of the idea that the interpretation for each predicate may vary under the context in which it occurs.

The definition of the interpreted universe is a unique formula in interpretations. We also flexibilize this requirement, allowing the context of occurrence to play a hole in what it means to make a quantification. Before, the quantifier $\forall x$ would turn into $\forall x(U(x) \rightarrow \dots)$ (abbreviation $\forall x \in U$), now, the quantifier will turn into $\forall x(U_\alpha(x) \rightarrow \dots)$ (abbreviation $\forall x \in U_\alpha$), where α is the quantified formula.

Let's now consider the actual way in which we may generate the translation. Before, we make a notational simplification that won't harm our result: we consider only prenex formulas. This, even though not detrimental, causes the need for extra proof: that any two prenex methods have the same effect in the translation.

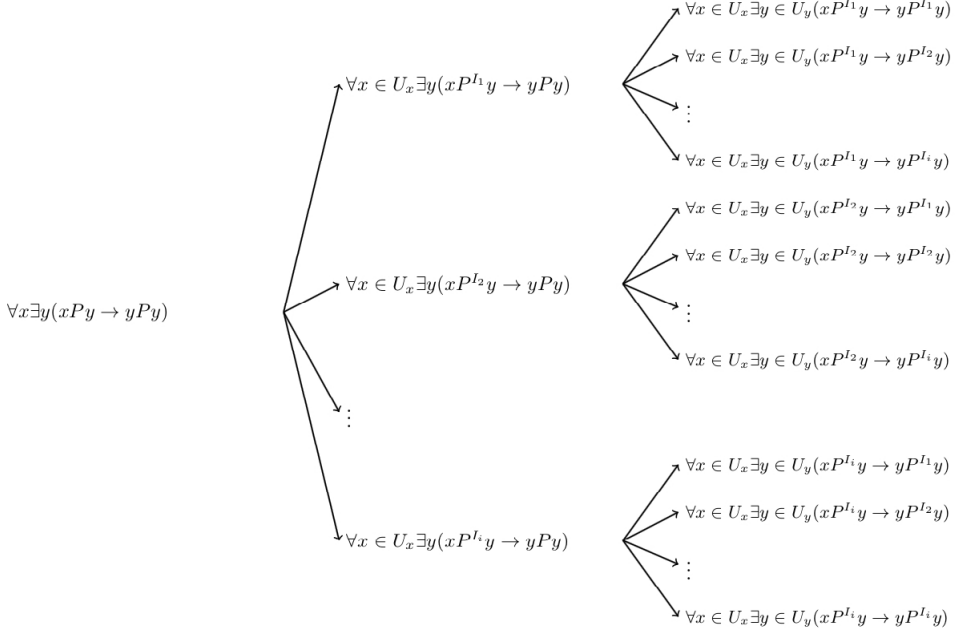
If in an interpretation we deal with each quantified variable in the order of occurrence, then the procedure for obtaining α^I from the sentence α will be finished in n (the number of quantifiers occurring in α) steps. For instance, if we take the formula $\forall x \exists y \forall z (xPy \wedge yPz \rightarrow xPz)$, the interpretation will occur in the following steps:

$$\begin{aligned} \forall x \in U \exists y \forall z (xP^I y \wedge yPz \rightarrow xP^I z), \\ \forall x \in U \exists y \in U \forall z (xP^I y \wedge yP^I z \rightarrow xP^I z), \\ \forall x \in U \exists y \in U \forall z \in U (xP^I y \wedge yP^I z \rightarrow xP^I z). \end{aligned}$$

In fact, the interpretation from α generates a sequence of n (number of quantifiers) steps $\alpha, \alpha_2, \alpha_3, \dots, \alpha_{n-1}, \alpha^I$. In contrast, the new method will form a tree of depth n . Each node in the tree will then ramify into all substitutions for the predicate.

Let's show a simplified illustration of the tree we may generate for the same formula $\forall x \exists y \forall z (xPy \wedge yPz \rightarrow xPz)$ (We abbreviate $U(x, \exists y(xPy \rightarrow yPy))$ and $U(y, (xPy \rightarrow yPy))$ for U_x and U_y). We will annotate by I_1, I_2, \dots, I_n each interpretation for the predicate P .

We follow with the precise definition of the interpretation tree. Before, some notation will be important to simplify our work.



For a binary predicate⁴ P in T_1 , we define a functor $I^P : \langle \beta_1, \beta_2 \rangle \longrightarrow \alpha$ and a set Id of two free variable (x_1 and x_2) formulas in T_2 . Since Id is an orderable set, we may denote $I^P \langle \varphi, \gamma \rangle$ by $P^{I(n,m)}$, being φ the n 'th formula in Id and γ the m 'th formula in Id . Finally, we call $x_i P^{I(n,m)} x_j$ the substitution of x_1 and x_2 for x_i and x_j in $P^{I(n,m)}$.

We define the transformation $*_{(x_i, I_k)}$ for open formulas:

1. If α is $x_i P x_j$, then $\alpha^{*(x_i, I_k)*_{(x_j, I_q)}}$ is $x'_i P^{I(k,q)} x'_j$.
2. If α is $x_i P x_j$, then $\alpha^{*(x_j, I_q)*_{(x_i, I_k)}}$ is $x'_i P^{I(k,q)} x'_j$.⁵
3. If x_i does not occur in α , then $\alpha^{*(x_i, I_k)}$ is α .
4. If α is $\gamma \vee \beta$, then $\alpha^{*(x_i, I_k)}$ is $\gamma^{*(x_i, I_k)} \vee \beta^{*(x_i, I_k)}$.
5. If α is $\neg \gamma$, then $\alpha^{*(x_i, I_k)}$ is $\neg \gamma^{*(x_i, I_k)}$.

The set Id fixes the uniformity of the treatment of the variables. Using the above structure, we may guarantee that each variable x is affected by a transformation of the type $*_{(x, I_k)}$ a single time.

⁴We will only consider the case in which we are dealing with theories with only a single binary predicate, for the other cases would follow easily.

⁵The purpose of having x_i substituted for x'_i guaranties that the transformation $*_{(x_i, I_k)}$ does not affect the variable x_i . Later, we will again use x_i as we can avoid double quantification using the variant equivalent formula. This will be sufficiently clear by the context.

The identity relation uses the same structure. The functor I^\equiv must internalize identity with the same properties as I^P . What fixes the relation between the predicates P and $=$ is the fact that they are bounded to the same set Id . By dealing with first-order logic with equality, we may need to impose some further conditions over I^\equiv . Further in the text, we will describe those conditions.

We follow with the definition of the translation tree:

Definition 2.1 *Let α be a prenex sentence of the form $Q_1x_1Q_2x_2\dots Q_nx_n\beta$ in T_1 , then $\alpha^{Tr(T_2)}$ is a tree such that α is the initial node and*

1. *each node of level i is of the form*

$$Q_1x_1 \in U_{x_1} \dots Q_ix_i \in U_{x_i} (Q_{i+1}x_{i+1} \dots Q_nx_n \beta^{*(x_1, I_{k_1})^*(x_2, I_{k_2}) \dots *(x_i, I_{k_i})}).$$
2. *the father node of the previous node is the node*

$$Q_1x_1 \in U_{x_1} \dots Q_{i-1}x_{i-1} \in U_{x_{i-1}} (Q_ix_i \dots Q_nx_n \beta^{*(x_1, I_{k_1})^*(x_2, I_{k_2}) \dots *(x_{i-1}, I_{k_{i-1}})}).$$
3. *and for each q this father node has a son $\beta^{*(x_1, I_{k_1})^*(x_2, I_{k_2}) \dots *(x_i, I_{k_i})}$.*

We say the translation is the generating process of this finite (or potentially infinite) tree. This is enough to guarantee that the translation is intelligible once we can write the tree as a precise and finite description (even in the potentially infinite case). However, we need to define validity in this tree in a way that satisfies the General Scheme of Interpretation and the Coherence Condition.

We then define the validity of the prenex sentences recursively. We do this in two stages: We define the condition of validity for the leaves and then the recursive way in which we should propagate the results of the leaves over the tree.

For those purposes, we define the $S : Form \longrightarrow Form$ as a functor that receives leaf formulas and returns a sentence that the regular proof system can evaluate:

Definition 2.2 (Leaf validity) *Let α be a prenex sentence in T_1 . If β is a leaf in $\alpha^{Tr(T_2)}$, then*

$$T_2 \vdash^s \beta \text{ if, and only if, } T_2 \vdash S(\beta)$$

Definition 2.3 (Node validity) *Let α be a prenex sentence in T_1 , then*

1. *If the quantification treated in β is universal, then $T_2 \vdash^s \beta$ if, and only if, $T_2 \vdash^s \gamma$ for each son of β .*

2. If the quantification treated in β is existential, then $T_2 \vdash^s \beta$ if, and only if, $T_2 \vdash^s \gamma$ for some son of β .

This definition still does not impose the necessary restrictions on satisfying the translation requirements. The reason for that, on the one hand, is that we haven't yet imposed restrictions over the predicates $x_i \in U_{x_i}$. On the other hand, we haven't imposed that the interpretation of equality satisfies versions of equality and identity axioms. However, we proceed with incomplete definitions, filling the gaps as it becomes necessary to prove the interpretation scheme.

2.1 Prenex condition

Up to this point, we only dealt with prenex sentences. Yet, we need a mechanism to bring formulas to their prenex equivalents in a organized way. We then proceed by expanding the definition of \vdash^s to formulas in general:

Definition 2.4 *Let α be any formula in T_1 , α' the prenex form of α and α'' the universal closure of α' , then $\alpha^{Tr(T_2)}$ is $(\alpha'')^{Tr(T_2)}$.*

With this definition, we have consequently defined validity in \vdash^s . It is enough to say $T_2 \vdash^s \alpha^{Tr(T_2)}$ if, and only if, $T_2 \vdash^s (\alpha'')^{Tr(T_2)}$. However, we should show that this definition preserves equivalence in the original theory. Two different and equivalent prenex formulas lead to the same result in \vdash^s system:

Definition 2.5 (Prenex condition of equivalence) *Let α_1 and α_2 be prenex forms of α .*

$$T \vdash^s (\alpha_1)^{Tr(T)} \iff T \vdash^s (\alpha_2)^{Tr(T)}$$

Initially, we analyze the simple case where α is $\forall x\beta \vee \exists y\gamma$, α_1 and α_2 are $\forall x\exists y(\beta \vee \gamma)$ and $\exists y\forall x(\beta \vee \gamma)$. For a simpler exposition, we take Id to be a set with only two elements.

In this case, we obtain two translation trees (we abbreviate $*(x, I_k)$ for $*_{x_k}$; the context makes it clear):

$$\alpha_1 := \forall x\exists y(\beta \vee \gamma) \begin{cases} \rightarrow \forall x \in U_x \exists y \in U_y (\beta^{*_{x_1}} \vee \gamma^{*_{x_1}}) \\ \rightarrow \forall x \in U_x \exists y \in U_y (\beta^{*_{x_2}} \vee \gamma^{*_{x_2}}) \end{cases} \begin{cases} \rightarrow \forall x \in U_x \exists y \in U_y (\beta^{*_{x_1}*_{y_1}} \vee \gamma^{*_{x_1}*_{y_1}}) \\ \rightarrow \forall x \in U_x \exists y \in U_y (\beta^{*_{x_1}*_{y_2}} \vee \gamma^{*_{x_1}*_{y_2}}) \\ \rightarrow \forall x \in U_x \exists y \in U_y (\beta^{*_{x_2}*_{y_1}} \vee \gamma^{*_{x_2}*_{y_1}}) \\ \rightarrow \forall x \in U_x \exists y \in U_y (\beta^{*_{x_2}*_{y_2}} \vee \gamma^{*_{x_2}*_{y_2}}) \end{cases}$$

and

$$\alpha_2 := \exists y \forall x (\beta \vee \gamma) \quad \begin{array}{l} \nearrow \\ \searrow \end{array} \quad \begin{array}{l} \exists y \in U_y \forall x (\beta^{*y_1} \vee \gamma^{*y_1}) \\ \exists y \in U_y \forall x (\beta^{*y_2} \vee \gamma^{*y_2}) \end{array} \quad \begin{array}{l} \nearrow \searrow \\ \nearrow \searrow \end{array} \quad \begin{array}{l} \exists y \in U_y \forall x \in U_x (\beta^{*y_1 * x_1} \vee \gamma^{*y_1 * x_1}) \\ \exists y \in U_y \forall x \in U_x (\beta^{*y_1 * x_2} \vee \gamma^{*y_1 * x_2}) \\ \exists y \in U_y \forall x \in U_x (\beta^{*y_2 * x_1} \vee \gamma^{*y_2 * x_1}) \\ \exists y \in U_y \forall x \in U_x (\beta^{*y_2 * x_2} \vee \gamma^{*y_2 * x_2}) \end{array}$$

Let's suppose that $T \vdash^s \alpha_1$. From the tree structure, this means

$$\{T \vdash \forall x \in U_x \exists y \in U_y (\beta^{*x_1 * y_1} \vee \gamma^{*x_1 * y_1}) \text{ or } T \vdash \forall x \in U_x \exists y \in U_y (\beta^{*x_1 * y_2} \vee \gamma^{*x_1 * y_2})\}$$

and

$$\{T \vdash \forall x \in U_x \exists y \in U_y (\beta^{*x_2 * y_1} \vee \gamma^{*x_2 * y_1}) \text{ or } T \vdash \forall x \in U_x \exists y \in U_y (\beta^{*x_2 * y_2} \vee \gamma^{*x_2 * y_2})\}$$

If we want to prove that $T \vdash^s \alpha_2$, we need to show this implies

$$\{T \vdash \exists y \in U_y \forall x \in U_x (\beta^{*y_1 * x_1} \vee \gamma^{*y_1 * x_1}) \text{ and } T \vdash \exists y \in U_y \forall x \in U_x (\beta^{*y_1 * x_2} \vee \gamma^{*y_1 * x_2})\}$$

or

$$\{T \vdash \exists y \in U_y \forall x \in U_x (\beta^{*y_2 * x_1} \vee \gamma^{*y_2 * x_1}) \text{ and } T \vdash \exists y \in U_y \forall x \in U_x (\beta^{*y_2 * x_2} \vee \gamma^{*y_2 * x_2})\}$$

From the definition of $T \vdash^s \alpha_1$ and because x/y does not occur in γ/β , we conclude

$$\{T \vdash \forall x \in U_x \beta^{*x_1} \vee \exists y \in U_y \gamma^{*y_1} \text{ or } T \vdash \forall x \in U_x \beta^{*x_1} \vee \exists y \in U_y \gamma^{*y_2}\}$$

and

$$\{T \vdash \forall x \in U_x \beta^{*x_2} \vee \exists y \in U_y \gamma^{*y_1} \text{ or } T \vdash \forall x \in U_x \beta^{*x_2} \vee \exists y \in U_y \gamma^{*y_2}\}$$

Similarly, we need to implicate:

$$\{T \vdash \forall x \in U_x \beta^{*x_1} \vee \exists y \in U_y \gamma^{*y_1} \text{ and } T \vdash \forall x \in U_x \beta^{*x_2} \vee \exists y \in U_y \gamma^{*y_1}\}$$

or

$$\{T \vdash \forall x \in U_x \beta^{*x_1} \vee \exists y \in U_y \gamma^{*y_2} \text{ and } T \vdash \forall x \in U_x \beta^{*x_2} \vee \exists y \in U_y \gamma^{*y_2}\}$$

Nonetheless, the implication we want is not the case. It is enough to notice the case

$$\begin{array}{l} T \vdash \forall x \in U_x \beta^{*x_1} \vee \exists y \in U_y \gamma^{*y_1} \\ T \not\vdash \forall x \in U_x \beta^{*x_1} \vee \exists y \in U_y \gamma^{*y_2} \\ T \not\vdash \forall x \in U_x \beta^{*x_2} \vee \exists y \in U_y \gamma^{*y_1} \\ T \vdash \forall x \in U_x \beta^{*x_2} \vee \exists y \in U_y \gamma^{*y_2} \end{array}$$

This case would satisfy $T \vdash^s \alpha_1$, but it won't satisfy $T \vdash^s \alpha_2$. In order to solve this problem, we should require that the proof of soundness for the

translation occur not in the theory T itself, but in some completion (deciding over undecidable formulas) of it that satisfies the desired implication.

We can, as we will do, bound the satisfaction relation to a **complete extension of \mathbf{T}** . In this case, the prenex condition is naturally satisfied. This means that our relation \vdash^s is relative to a choice of completion over the translating theory: the s in \vdash^s represents the completion strategy over the original theory and the functor S . We need therefore to amend the condition over the leaves in \vdash^s :

Definition 2.6 (Amended) *Let α be a prenex sentence in T_1 and T_2^s a completion of T_2 , then*

1. *If β is a leaf in $\alpha^{Tr(T_2)}$, then $T_2 \vdash^s \beta$ if, and only if, $T_2^s \vdash S(\beta)$.*
2. *If the quantification treated in β is universal, then $T_2 \vdash^s \beta$ if, and only if, $T_2 \vdash^s \gamma$ for each son of β .*
3. *If the quantification treated in β is existential, then $T_2 \vdash^s \beta$ if, and only if, $T_2 \vdash^s \gamma$ for some son of β .*

Yet, the required condition could be slightly weaker than taking T^s to be complete. We need only to have the completion over the formulas in the languages of the leaves in \vdash^s :

Definition 2.7 (prenex equivalence sub-condition) *$Tr(T)$ in \mathcal{L} and T^s are such that, for all α in \mathcal{L} , if $T \vdash^s \alpha^{Tr(T)}$, then, for every leaf α^{*i} in the tree $\alpha^{Tr(T)}$,*

$$T^s \vdash S(\alpha^{*i}) \text{ or } T^s \vdash S(\neg\alpha^{*i})$$

To simplify the proofs in this section, we consider S to be the identity functor. The proof of the following lemma should be similar for any adequate functor S :

Lemma 2.8 *If $Tr(T)$ in \mathcal{L} and T satisfies the prenex equivalence sub-condition, then they satisfy the prenex equivalence condition.*

Proof. We only sketch the proof, showing that the sub-condition is enough to overcome the difficulty presented above.

Suppose $T \vdash^s \alpha_1^{Tr(T)}$, then

$$\{T \vdash \forall x \in U_x \beta^{*x_1} \vee \exists y \in U_y \gamma^{*y_1} \text{ or } T \vdash \forall x \in U_x \beta^{*x_1} \vee \exists y \in U_y \gamma^{*y_2}\}$$

$$\text{and } \{T \vdash \forall x \in U_x \beta^{*x_2} \vee \exists y \in U_y \gamma^{*y_1} \text{ or } T \vdash \forall x \in U_x \beta^{*x_2} \vee \exists y \in U_y \gamma^{*y_2}\}$$

As T is complete for leaf formulas in $\alpha_1^{Tr(T)}$, we obtain

$$\{T \vdash \forall x \in U_x \beta^{*x_1} \text{ or } T \vdash \exists y \in U_y \gamma^{*y_1} \text{ or } T \vdash \forall x \in U_x \beta^{*x_1} \text{ or } T \vdash \exists y \in U_y \gamma^{*y_2}\}$$

and

$$\{T \vdash \forall x \in U_x \beta^{*x_2} \text{ or } T \vdash \exists y \in U_y \gamma^{*y_1} \text{ or } T \vdash \forall x \in U_x \beta^{*x_2} \text{ or } T \vdash \exists y \in U_y \gamma^{*y_2}\}$$

This tautologically implies

$$\{(T \vdash \forall x \in U_x \beta^{*x_1} \text{ or } T \vdash \exists y \in U_y \gamma^{*y_1}) \text{ and } (T \vdash \forall x \in U_x \beta^{*x_2} \text{ or } T \vdash \exists y \in U_y \gamma^{*y_1})\}$$

$$\{(T \vdash \forall x \in U_x \beta^{*x_1} \text{ or } T \vdash \exists y \in U_y \gamma^{*y_2}) \text{ and } (T \vdash \forall x \in U_x \beta^{*x_2} \text{ or } T \vdash \exists y \in U_y \gamma^{*y_2})\}$$

Therefore

$$\{T \vdash \forall x \in U_x \beta^{*x_1} \vee \exists y \in U_y \gamma^{*y_1} \text{ and } T \vdash \forall x \in U_x \beta^{*x_2} \vee \exists y \in U_y \gamma^{*y_1}\}$$

$$\{T \vdash \forall x \in U_x \beta^{*x_1} \vee \exists y \in U_y \gamma^{*y_2} \text{ and } T \vdash \forall x \in U_x \beta^{*x_2} \vee \exists y \in U_y \gamma^{*y_2}\}$$

Thus, $T \vdash^s \alpha_2^{Tr(T)}$. By simple induction, we can finish the proof. What is left in the proof can be done by simple induction. \blacksquare

One may see the sub-condition as excessively arbitrary. This shouldn't be a problem; We hold as a principle that any explicit method of realizing a translation implies arbitrariness. The expressiveness of any technique is grounded in admitting restrictions and, in this sense, to fix some less compromising restrictions enlarge our capability of realizing translations. The only way to not impose restrictions is to not posit any method whatsoever.

Despite that, we hold that the sub-condition is not unmotivated. When, for example, we take the regular case of interpretation; We observe that the condition holds trivially. If $T \vdash \alpha^I$, then $T \vdash \alpha^I$ or $T \vdash \neg \alpha^I$. For \vdash^s , each translated "theorem" has a fully comprehensible translation tree in T . Naturally, the opposite may also be admissible: it is reasonable to hold that the understanding of a translated sentence depends on the full evaluation of the tree. A yet more flexible system of translation can be defined. However, we fix this level of restriction, for it is enough to account for the model theoretical proofs of relative consistency.

2.2 Coherence condition

As stated before, we need to prove the connection with the concept of relative consistency. Thus, we prove:

Theorem 2.9 (for coherence condition) *If $T \vdash^s \alpha^{Tr(T)}$ and $T \vdash^s (\neg \alpha)^{Tr(T)}$, then there is β such that $T^s \vdash \beta$ and $T^s \vdash \neg \beta$.*

Proof. We note that, if a prenex formula α is of the form $\overline{Qx}(\alpha')$, then there is another for $\neg\alpha$ of the form $\overline{Q'x}(\neg\alpha')$ (Being Q' the dual quantification for Q). Thus, in each level of the translation tree $\alpha^{Tr(T)}$ and $\neg\alpha^{Tr(T)}$, we are dealing with one universal quantification and one existential quantification. Moreover, we observe that the leaves in both trees are organized similarly. If we overlap the trees, the superposed leaves will be such that: if $\overline{Qx} \in \overline{U_x}(\alpha^*)$ is the leaf of one tree, then $\overline{Q'x} \in \overline{U_x}(\neg\alpha^*)$ is the leaf of the other (in this case we say the leaves have the same position).

In view of these remarks, we prove by induction that if $T \vdash^s \alpha^{Tr(T)}$ and $T \vdash^s (\neg\alpha)^{Tr(T)}$, then at least one leaf $\overline{Qx} \in \overline{U_x}(\alpha^*)$ in $\alpha^{Tr(T)}$ and the leaf of same position $\overline{Q'x} \in \overline{U_x}(\neg\alpha^*)$ in $(\neg\alpha)^{Tr(T)}$ are such that $T^s \vdash \overline{Qx} \in \overline{U_x}(\alpha^*)$ and $T^s \vdash \overline{Q'x} \in \overline{U_x}(\neg\alpha^*)$. Naturally, $\overline{Q'x} \in \overline{U_x}(\neg\alpha^*)$ is logically equivalent to $\neg(\overline{Qx} \in \overline{U_x}(\alpha^*))$, therefore, if the result holds, there is a formula $\beta \equiv \overline{Qx} \in \overline{U_x}(\alpha^*)$ such that $T^s \vdash \beta$ and $T^s \vdash \neg\beta$ as wanted.

By induction, suppose that at some level k of the tree it holds for two nodes of same position

$$T \vdash^s Q_1x_1 \in U_{x_1} \dots Q_kx_k \in U_{x_k} \overline{Qx}(\alpha^{*(1,2,\dots,k)}) \quad (1)$$

$$T \vdash^s Q'_1x_1 \in U_{x_1} \dots Q'_kx_k \in U_{x_k} \overline{Q'_x}(\neg\alpha^{*(1,2,\dots,k)}), \quad (2)$$

being $*_{\langle 1,2,\dots,k \rangle}$ the abbreviation for some sequence of the form $*(x_1, I_{d_1}) *(x_2, I_{d_2}) \dots *(x_k, I_{d_k})$.

For the next level, at least one of the quantifiers Q_{k+1} or Q'_{k+1} is universal. Suppose, without loss of generality, Q_{k+1} is universal; Here Q'_{k+1} is existential. For it holds the equation 2 above and Q'_{k+1} is existential, we know that, for some son $*_{\langle 1,2,\dots,k,k+1 \rangle}$ of node $*_{\langle 1,2,\dots,k \rangle}$,

$$T \vdash^s Q'_1x_1 \in U_{x_1} \dots Q'_kx_k \in U_{x_k} Q'_{k+1}x_{k+1} \in U_{x_{k+1}} \overline{Q'_x}(\neg\alpha^{*(1,2,\dots,k,k+1)}). \quad (3)$$

For any son γ_i of $*_{\langle 1,2,\dots,k \rangle}$ in $\alpha^{Tr(T)}$, $T \vdash^s \gamma_i$. In particular,

$$T \vdash^s Q_1x_1 \in U_{x_1} \dots Q_kx_k \in U_{x_k} Q_{k+1}x_{k+1} \in U_{x_{k+1}} \overline{Qx}(\alpha^{*(1,2,\dots,k,k+1)}). \quad (4)$$

as desired.

As the case where $k = 0$ is equivalent to the supposition of the theorem, then this argument finishes the induction. ■

2.3 Preserving First Order Logic derivation

We prove that the system \vdash^s preserves logic rules and axioms. From that, the interpretation scheme will follow naturally. We start by proving the following fact over tautologies:

Lemma 2.10 *If α is a tautology, then $\vdash^s \alpha$.*

Proof. The proof is a natural consequence of the fact that $*(x_1, i_1) *(x_2, i_2) \dots *(x_n, i_n)$ preserves the boolean structure of the formulas. If the original formula is a tautology, then each leaf of the translation tree is a tautology. ■

Now we should specify the conditions for the identity operation. The first aim in this case is to recover the identity axiom:

Lemma 2.11 *If α is an axiom of identity $x = x$, then $\vdash^s \alpha^{Tr(T)}$.*

The translation for $(x = x)^{Tr(T)}$ has leaves of the form $\forall x \in U_x(x =^{I(k,k)} x)$; thus we need for every k

$$T \vdash \forall x \in U_x(I^{\langle \alpha_k, \alpha_k \rangle}(x, x))$$

For this purpose, we add the first restriction on the functor $I^{\langle \cdot, \cdot \rangle}$:

Definition 2.12 (Identity condition) *The functor $I^{\langle \cdot, \cdot \rangle}$ in $Tr(T)$ should be such that $I^{\langle \alpha_k, \alpha_k \rangle}(x, x)$ is a quasi-tautology.*

The axiom of identity must be valid in any context. We should have a formula U which encapsulates all possible contexts $U(x, \alpha)$.

Definition 2.13 (Condition over the universe of quantification) *There is a formula U such that, for every α , $T \vdash x \in U(x, \alpha) \rightarrow x \in U$.*

With this, we may fix the context for identity $U(x, x = x) = U$. We follow with the evaluation of the equality axiom:

Lemma 2.14 *If α is a equality axiom:*

1. $x = y \rightarrow f(x) = f(y)$
2. $x = y \wedge z = w \rightarrow (xPz \leftrightarrow yPw)$

then $\vdash^s \alpha^{Tr(T)}$.

Notably, if we have an axiom of the form $x = y \wedge z = w \rightarrow x \in z \leftrightarrow y \in w$, then the leaves in the translation tree have the form:

$$\forall x, y, z, w \in U_{x,y,z,w}((x =^{I(i,j)} y) \wedge (z =^{I(k,q)} w) \rightarrow (x =^{P(i,k)} z) \leftrightarrow (y =^{P(j,q)} w))$$

Here we insert one more restriction for the identity functor; It must not depend on the structure of the predicate P . However, as the functor I^P is too comprehensive, the restriction insides heavily over the identity. Ultimately, this makes the formulas over x and y to be inter-substitutable.

The same argument over the quantifier context applies to equality. For this reason, the abbreviation $\forall x, y, z, w \in U_{x,y,z,w}$ must be $\forall x, y, z, w \in U$.

Definition 2.15 (equality condition) *The functor $I^=$ is such that*

$$\vdash I^= \langle \alpha_i, \alpha_j \rangle (x, y) \rightarrow \forall z \in U(\alpha_i(z) \leftrightarrow \alpha_j(z))$$

This condition is, naturally, compatible with the identity condition, once it is a tautological consequence of the equality condition in case $i = j$. It remains to show that the lemma follows from the condition on identity:

Proof. [2.14]

We show that, for every i, j, k, q , if the condition holds, then

$$\forall x, y, z, w \in U((x =^{I(i,j)} y) \wedge (z =^{I(k,q)} w) \rightarrow (x =^{P(i,k)} z) \leftrightarrow (x =^{P(i,k)} z)).$$

Suppose $(x =^{I(i,j)} y) \wedge (z =^{I(k,q)} w)$, then, we obtain

$$\forall a \in U(\alpha_i(a) \leftrightarrow \alpha_j(a)) \wedge \forall a \in U(\alpha_k(a) \leftrightarrow \alpha_q(a))$$

From this, we concludes, from equality theorem,

$$\forall x, z \in U(I^P \langle \alpha_i, \alpha_k \rangle (x, z) \leftrightarrow I^P \langle \alpha_j, \alpha_q \rangle (x, z)).$$

This, in turn, implies

$$\forall x, z, y, w \in U(I^P \langle \alpha_i, \alpha_k \rangle (x, z) \leftrightarrow I^P \langle \alpha_j, \alpha_q \rangle (y, w)).$$

■

We continue with the proof for the substitution axiom for logic.

Lemma 2.16 *If α is a substitution axiom, $\vdash^s \alpha^{Tr(T)}$.*

We now need to define the treatment for the constants. Here, we fix a formula in Id for each constant and then treat them as we do with variables:

1. c^I is some $\alpha \in Id$.
2. if α is $x_i P c$, then $\alpha^{*(x_i, I_k)}$ is $I^P \langle \alpha_i, c^I \rangle (x'_i, x_c)$.
3. if α is $c P x_i$, then $\alpha^{*(x_i, I_k)}$ is $I^P \langle c^I, \alpha_i \rangle (x_c, x'_i)$.

This will be well defined if the following condition holds:

Definition 2.17 (Condition for constants) *For every constant*

$$T \vdash^s (\exists! x(x = c))^{Tr(T)}.$$

We note that we can emulate the transformation for the constants as any variable. The single important difference is that, for the constants, the translation tree does not ramify. For this reason, we introduce the notation: $*(x, I_c)$ which is equivalent to $*(x, I_i)$, when the i 'th formula of Id is the formula for the constant c .

We follow with the proof for the lemma.

Proof. An axiom of substitution $\gamma(c) \rightarrow \exists x\gamma(x)$ has as prenex form a formula like $\exists x\overline{Qy}(\beta(c, \overline{y}) \rightarrow \beta(x, \overline{y}))$. It follows that the leaf for the translation tree is of the form:

$$\exists x \in U_x \overline{Qy} \in U_y (\beta^{*(x, I_c)* (y_1, I_{k_1}) \dots * (y_n, I_{k_n})} \rightarrow \beta^{*(x, I_{k_0}) * (y_1, I_{k_1}) \dots * (y_n, I_{k_n})}) \quad (5)$$

We call the left side of the previous formula β_1 and the right side β_2 . If we take the ramification of the first level where $I_{k_0} = I_c$, we observe that all leaves in this ramification are such that $\beta_1 = \beta_2$. Notably, this means that all leaves are tautologies for this branch. Thus, if $I_i = I_c$

$$T \vdash^s \exists x \in U_x (\gamma(c) \rightarrow \gamma(x))^{*(x, I_i)}$$

This, in turn, is enough to obtain the lemma, since the quantification over x is existential. ■

Lemma 2.18 (Modus Ponens) *If $T \vdash^s \alpha$ and $T \vdash^s \alpha \rightarrow \beta$, then $T \vdash^s \beta$.*

Proof. Let's pay special attention to the translation tree for $\alpha \rightarrow \beta$. Notably, the prenex form may assume many equivalent formats. To facilitate the proof, we chose the prenex steps conveniently:

Prenex operation:

One substitute

1. α for a variant of α .
2. $\neg\forall x\alpha$ for $\exists x\neg\alpha$.
3. $\neg\exists x\alpha$ for $\forall x\neg\alpha$.
4. $Qx(\alpha) \vee \beta$ for $Qx(\alpha \vee \beta)$ (valid only if x does not occur free in β).
5. $\alpha \vee Qx(\beta)$ for $Qx(\alpha \vee \beta)$ (valid only if x does not occur free in α).

We do the prenexation of $\alpha \rightarrow \beta$ ($\neg\alpha \vee \beta$) following the steps:

1. we obtain the prenex form of $\neg\alpha$ and β separately, obtaining $\overline{Q_\alpha x}((\neg\alpha)') \vee \overline{Q_\beta y}((\beta)')$ (We may need to apply the operation 1 sometimes to avoid that the quantifications in each formula are the same).

2. we apply 5 to extract all quantifications $\overline{Q_{\beta y}}$.
3. we apply 4 to extract all quantifications $\overline{Q_{\alpha x}}$.

Using those operations, we obtain:

$$\overline{Q_{\alpha x}} \overline{Q_{\beta y}}((\neg\alpha)' \vee (\beta)') \quad (6)$$

We thus note that the transformations in the translation tree are going to affect $(\beta)'$ only after all transformations that affect $(\neg\alpha)'$ were applied.

Because of that, the trees for α and $\neg\alpha \vee \beta$ become similar up to the ending level of α (level k). On the other hand, the branches from this level in $\neg\alpha \vee \beta$ are structurally similar to the tree of β .

The level k node in $\neg\alpha \vee \beta$ has the following form:

$$\overline{Q_{\beta y}}(\neg\alpha^* \vee \beta) \quad (7)$$

However, from the same procedure as in 2.9, it must be a level k node such that

$$T \vdash^s \overline{Q_{\beta y}}(\neg\alpha^* \vee \beta) \quad (8)$$

and

$$T \vdash^s \alpha^* \quad (9)$$

Let's then analyze the branch for $\alpha \rightarrow \beta$ from the node $\overline{Q_{\beta y}}(\neg\alpha^* \vee \beta)$.

Notably, this branch is s-valid only if T^s proves the formula of the form $(\neg\alpha^* \vee \beta^{*'})$ in the same structural way as the validity for the β tree itself. That is, if for each $*'$, T^s proves $(\neg\alpha^* \vee \beta^{*'})$ and also $(\beta^{*'})$, then $T \vdash^s \beta$.

In fact, as $T \vdash \alpha^*$, then for each $*'$ such that $T \vdash (\neg\alpha^* \vee \beta^{*'})$. Thus $T \vdash \beta^{*'}$ by *modus ponens*. Therefore, $T \vdash^s \beta$ as wanted. ■

It is only need then to prove the rule for quantifier introduction:

Lemma 2.19 *If $T \vdash^s \alpha \rightarrow \beta$, and x does not occur free in β , then $T \vdash^s \exists x \alpha \rightarrow \beta$.*

Proof. Since x is a free variable in α , then it is not in β . We can without loss arbiter that the first level in the tree for $\alpha \rightarrow \beta$ refers to the free variable x quantified universally. Thus, we obtain $T \vdash^s \alpha^{*1} \rightarrow \beta$ for every $*_1$.

From definition $T \vdash^s \exists x \alpha \rightarrow \beta$ in case $T \vdash^s \alpha^{*1} \rightarrow \beta$ for some $*_1$. And this is naturally the case since it is already true for all $*_1$. ■

This finishes the logical treatment of the model-style translation. And thus we can say that, if $Tr(T_2)$ and \vdash^s satisfies all conditions described in this section and $T_2 \vdash^s \alpha^{Tr(T_1)}$ for all $\alpha \in T_1$, then it satisfies the **general scheme of interpretation** and the **coherence condition**.

3 Defining the translation of NBG in ZF

To define the translation of NBG in ZF, we are going to rely on Novak's strategy given in [3]. From a complete Henkin extension ZF^s of ZF, we define the countable model \mathcal{M} for NBG:

1. Let \sim be the equivalence relation of one free variable formulas in ZF^s :
 $\alpha \sim \beta \equiv \forall x(\alpha(x) \leftrightarrow \beta(x))$. We denote the equivalence class of a formula by $\ulcorner \alpha \urcorner$.
2. M is the set of all equivalence classes $\ulcorner \alpha \urcorner$.
3. $\ulcorner \alpha \urcorner \in^M \ulcorner \beta \urcorner$ if, and only if, there is a constant c in ZF^s such that $\forall x(\alpha(x) \leftrightarrow x \in c)$ and $\beta(c)$ are in ZF^s .
4. Every Henkin constant c denotes $\ulcorner x \in c \urcorner$.

From this definition, one can prove that $\mathcal{M} \models NBG$. Our goal then is to show that we can emulate this model construction in the translating system described in previous sections.

The main difficulty is to emulate the Henkin constants introduced in ZF^s . We first use the index of the variables to arrange the set Id in our favor: Let $GN(\alpha)$ be the Gödel numbering of the formula α , then

$$n(\alpha) = \min\{GN(\beta) \mid ZF^s \vdash \forall x(\beta(x) \leftrightarrow \alpha(x))\} \quad (10)$$

Let special indexes be the set of $n(\alpha)$'s. From this we can devise the special variables as $y_{n(\alpha_1)}, y_{n(\alpha_2)}, \dots$ and the normal variables x_1, x_2, x_3, \dots . We now define our Id set:

$$\gamma \in Id \text{ if, and only if, } \gamma \text{ has only one normal free variable.} \quad (11)$$

We define the functor for membership relation:

$$I^\in \langle \alpha, \beta \rangle = \exists z(\forall w(\alpha(w) \leftrightarrow w \in z) \wedge \beta(y_{n(\alpha)})) \quad (12)$$

Basically, the functor is saying that α is a formula that stands for a set and that the special variable for α is a "member" of β . Notably, we now have a free variable $y_{n(\alpha)}$ that needs specification. We have bounded every occurrence of

equivalent formulas to a single special variable – thus, once we force $y_{n(\alpha)}$ to be exactly the set whose membership coincides with α , then those equivalent occurrences will be bounded to the same variable.

We specify the special variables in the final functor for \vdash^s :

$$\begin{aligned} S(\beta) = & \exists w \forall z (\alpha_1(z) \leftrightarrow z \in w) \rightarrow y_{n(\alpha_1)} = w \wedge \\ & \exists w \forall z (\alpha_2(z) \leftrightarrow z \in w) \rightarrow y_{n(\alpha_2)} = w \wedge \\ & \vdots \\ & \wedge \exists w \forall z (\alpha_k(z) \leftrightarrow z \in w) \rightarrow y_{n(\alpha_k)} = w \rightarrow \beta \end{aligned}$$

where $y_{n(\alpha_1)}, y_{n(\alpha_2)}, \dots, y_{n(\alpha_k)}$ are the special variables occurring in β .

This trick using the special variables is enough to emulate the Henkin constants in Novak’s technique. It is routine to verify that this S functor won’t affect the conditions exposed in the previous section.

The problem here is that we do not use the full strength of the method. Basically, we are solely using the branching of the translation tree to insert the canonical construction for a model. Yet, we have shown a basis for attributing translating meaning to a particular method of model theoretical reduction. This method can be applied more generally to every model construction that uses definable classes over Henkin canonical construction – and this can be done in the same spirit as described above.

4 Combining interpretations

Philosophers and mathematicians use extensively interpretations in the study of reductions among theories. Yet, we may find difficulty in arguing for a particular interpretation being **the intended reduction**.

When we provide an interpretation I of a theory T_1 into a theory T_2 , we require for each $\alpha \in T_1$ that $T_2 \vdash \alpha^I$. What may be the case (and often is) is that T_2 proves β^I for many undecidable formulas of T_1 . Model-theoretically, this means that the interpretation in T_2 is discarding many possible models for T_1 . Naturally, T_2 may eliminate only bad models for T_1 , however it may be the case that T_2 is eliminating precisely the intended model for T_1 .⁶

⁶It is widely accepted that the ordinal interpretation in ZFC for the natural numbers is the intended model for arithmetics (even though we might not have formal criteria to determine that). Nonetheless, the determines of this model still depends on deciding undecidable formulas in ZFC. Thus in coordinating the decision over undecidable formulas of ZFC and of PA, we may end up with a mismatch between the extension of PA and the interpretation of PA in the extension of ZFC.

By using the enlarged concept of interpretation, we want to minimize the number of possible models we exclude in a standard interpretation. The method we will explore in this section is combining interpretations. In this case, we change the requirement over the open formulas in $*(x_i, J_k)$:

Each composition of $*(x_i, J_k)$ should lead to a particular interpretation for the predicates we already have for the languages of the theories in question $\{I_0, I_1, \dots, I_{n-1}\}$. Any method suffices. We define here a particular type:

1. If α is $\gamma \vee \beta$, then $\alpha^{*(x_i, J_k)}$ is $\gamma^{*(x_i, J_k)} \vee \beta^{*(x_i, J_k)}$.
2. If α is $\neg\gamma$, then $\alpha^{*(x_i, J_k)}$ is $\neg\gamma^{*(x_i, J_k)}$.

If the formula in question is a leaf, we start generating the correspondent interpretation:

Let $a_i \in \{0, 1\}$ and $(a_1 a_2 \dots a_q)_2 \bmod(n)$ the remainder in the division by n of the decimal representation of the binary representation $a_1 a_2 \dots a_q$.

Let γ be an atomic formula and $k = (k_1 k_2 \dots k_q)_2 \bmod(n)$, then

$$\gamma^{*(x_i, J_{k_1})^{*(x_i, J_{k_2})} \dots^{*(x_i, J_{k_q})} \text{ is } \gamma^{I_k} \quad (13)$$

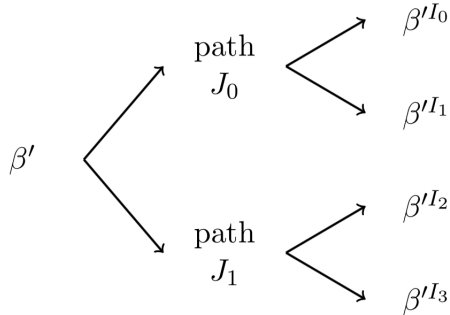
We now explore an illustrative example:

Let $T_1 = \{\forall x \exists y \alpha, \exists x \forall y \beta, \exists x \forall y \gamma\}$ and say I_0 and I_2 are interpretations of T_1 in T_2 . Moreover,

I_1 interprets in T_2 the formulas $\{\forall x \exists y \alpha, \exists x \forall y \beta, \neg \exists x \forall y \gamma\}$

I_3 interprets in T_2 the formulas $\{\forall x \exists y \alpha, \neg \exists x \forall y \beta, \exists x \forall y \gamma\}$

Suppose all those interpretations use the same universe of interpretation U , let $*(x_i, J_k)$ be written as $*_k$ and take Id to be $\{J_0, J_1\}$. We note that $*_0*_0 = I_0$, $*_0*_1 = I_1$, $*_1*_0 = I_2$ and $*_1*_1 = I_3$. We evaluate the translation tree for $\beta' = \forall x \exists y \beta$:



Since $T_2 \vdash \beta^{I_0}$ and $T_2 \vdash \beta^{I_1}$, then the path 0 is valid universally – in turn, the initial node is valid existentially. Therefore, $T_2 \vdash^s \beta^{Tr(T_2)}$. Similarly, we can conclude that $T_2 \vdash^s \alpha^{Tr(T_2)}$ and $T_2 \vdash^s \gamma^{Tr(T_2)}$. Finally $T_2 \vdash^s T_1^{Tr(T_2)}$.

Thus, we have a new translation method for T_1 in T_2 using the interpretations I_0, I_1, I_2 and I_3 . Nonetheless, only I_0 and I_2 are interpretations of the original theory. We have allowed some flexibility to the translation requirements that make it possible to use partial interpretations I_3 and I_4 significantly.

Furthermore, we still can push it a little further redefining I_0 and I_2 :

I_0 interprets in T_2 the formulas $\{\neg\forall x\exists y\alpha, \exists x\forall y\beta, \neg\exists x\forall y\gamma\}$

I_2 interprets in T_2 the formulas $\{\neg\forall x\exists y\alpha, \exists x\forall y\beta, \exists x\forall y\gamma\}$

Now we are in a situation in which none of the I 's are interpretations of T_1 in T_2 . But for we still have $T_2 \vdash^s T_1^{Tr(T_2)}$ all of them together can account for a translation of T_1 in the enlarged sense of translation.

As we have discussed before, this result should still imply the relative consistency of the theories. Therefore, this scheme shows that we can indeed use a plurality of partial interpretations to prove a result of relative consistency. This may be true even if the theory in question is not interpretable in the translating theory.

5 Final remarks

We have worked with a flexible type of translation, allowing many changes in the structure of the traditional interpretation. By doing so, we could include as a translation the model theoretical proof of relative consistency between ZF and NBG. Furthermore, we have shown that this enlarged concept of translation can reach new conditions of reducibility for models excluded by excessive requirements of uniformity.

In order to accomplish these relations, we have described the requirements of **General scheme of interpretation** and the **Coherence condition**. Subsequently, we allowed ourselves to use in our favor what was left undefined. Yet, we haven't explored the full expressive power of this method. An account of what are the models we can indeed construct is still needed. Thus, further development should account for the new truth conditions we can describe for a given theory as PA, ZF, and others.

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